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**Abstract**

We introduce the funnel observer as a novel and simple adaptive observer of "high-gain type". We show that this observer is feasible for a large class of nonlinear systems described by functional differential equations which have a known strict relative degree, the internal dynamics map bounded signals to bounded signals, and the operators involved are sufficiently smooth. Apart from that the funnel observer does not need specific knowledge of the system parameters, and we show that it guarantees prescribed transient behavior of the observation error. We compare the funnel observer to existing (adaptive) high-gain observers and illustrate it by a simulation of a bioreactor model.

*Keywords:* nonlinear systems; funnel observer; observer design; high-gain observer.

## 1. Introduction

In the present paper we propose a novel and simple adaptive observer of "high-gain type", the *funnel observer*. The high-gain parameter is determined adaptively online such that the observer output error satisfies a prescribed transient behavior.

High-gain observers have been developed around 30 years ago in the works [2, 13, 15, 17], see also the recent survey [12]. Choosing the observer gain $k$ large enough, the observer error can be made arbitrarily small, see e.g. [18]. The advantage of high-gain observers is that they can be used to estimate the system states without knowing the exact parameters; only some structural assumptions, such as a known relative degree, are necessary. Furthermore, they are robust with respect to input noise. The drawback is that it is not known a priori how large $k$ must be chosen and appropriate values must be identified by offline simulations. If $k$ is chosen unnecessarily large, the sensitivity to measurement noise increases dramatically.

In order to resolve these problems, the constant high-gain parameter $k$ has been replaced by an adaptation scheme in [1]. The gain $k(t)$ is determined by a differential equation depending on the observation error. This leads to a monotonically increasing $k(t)$ as long as the observation error lies outside a predefined $\lambda$-strip $[-\lambda,\lambda]$, and it stops increasing as soon as the error enters the strip. The advantage of this observer is that $k(t)$ is adapted online to the actual needed value, which also leads to lower high-gain parameters in general. However, $k(t)$ is monotonically non-decreasing and hence susceptible to unwarranted increase due to perturbations to the system. Furthermore, while convergence of the observation error to the $\lambda$-strip is guaranteed, its transient behavior cannot be influenced.

To resolve these issues we introduce the following funnel observer:

$$\boxed{\begin{aligned}
\dot z_1(t) &= z_2(t) + \big(q_1 + p_1 k(t)\big)\big(y(t)-z_1(t)\big),\\
\dot z_2(t) &= z_3(t) + \big(q_2 + p_2 k(t)\big)\big(y(t)-z_1(t)\big),\\
&\;\vdots\\
\dot z_{r-1}(t) &= z_r(t) + \big(q_{r-1} + p_{r-1} k(t)\big)\big(y(t)-z_1(t)\big),\\
\dot z_r(t) &= \tilde\Gamma u(t) + \big(q_r + p_r k(t)\big)\big(y(t)-z_1(t)\big),\\
k(t) &= \frac{1}{1-\varphi(t)^2\|y(t)-z_1(t)\|^2},
\end{aligned}} \tag{1}$$

where the design parameters $p_i>0$, $q_i>0$, $\tilde\Gamma\in\mathbb{R}^{m\times m}$ and the function $\varphi:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ are explained in detail in Section 3.

We like to emphasize that:

- The proposed adaptation scheme for $k(t)$ is simple, non-dynamic, and non-monotone,
- it guarantees prescribed transient behavior of the observation error, and
- all advantages of high-gain observers (e.g., only little knowledge of the system required, excellent robustness properties) are retained.

To illustrate the observer (1) we consider, as a prototype, the following minimum-phase linear time-invariant system

$$\begin{aligned}
\dot x(t) &= Ax(t) + Bu(t),\\
y(t) &= Cx(t)
\end{aligned} \tag{2}$$

with strict relative degree $r\in\mathbb{N}$, i.e., $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$ and $C\in\mathbb{R}^{m\times n}$ with the properties:

(A1) $\operatorname{rk}_{\mathbb{C}}\begin{bmatrix}\lambda I - A & B\\ C & 0\end{bmatrix} = n+m$ for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda\ge 0$;

(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B\in\mathbf{Gl}_m(\mathbb{R})$.

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Condition (A1) characterizes the minimum-phase assumption and condition (A2) the strict relative degree. For illustrative purposes the following theorem is not formulated in a mathematical rigorous way; we refer to our main result in Theorem 4.1.

**Theorem 1.1.** *Let $(x,u,y)$ be a solution of system* (2) *such that $y,\dots,y^{(r-1)}$ are bounded. Then the funnel observer* (1) *has an absolutely continuous and bounded solution $(z_1,\dots,z_r)$ such that k is bounded and*

$$\forall t>0:\ \varphi(t)\|y(t)-z_1(t)\|<1. \tag{3}$$

The proof is a consequence of Theorem 4.1.

We stress that condition (3) means prescribed transient behavior of the observation error $y(t)-z_1(t)$ in the sense that it is pointwise below a given funnel function $1/\varphi$, see Figure 1. To achieve this, the observer gain will be increased whenever $\|y(t)-z_1(t)\|$ approaches the funnel boundary. High values of the gain function lead to a faster decay of the observation error.

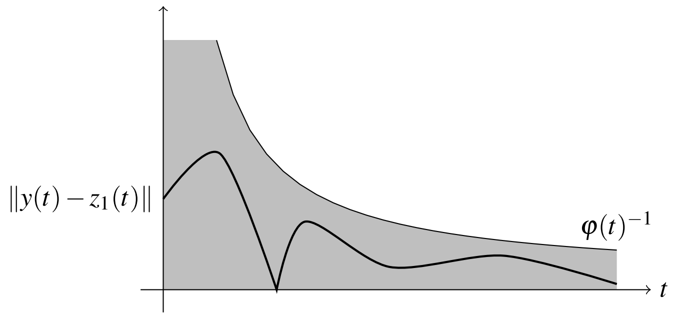


Figure 1: Observation error and funnel function

The funnel observer is not limited to linear systems (2). We show that the funnel observer (1) is feasible for a large class of nonlinear systems described by functional differential equations which satisfy that

(i) the system has known strict relative degree $r$,

(ii) the internal dynamics map bounded signals to bounded signals,

(iii) the operators involved are sufficiently smooth to guarantee local maximal existence of solutions.

The present paper is organized as follows: In Section 2 we specify the considered system class and discuss several important subclasses. The funnel observer is introduced in Section 3 and feasibility is proved in Section 4. A simulation of the funnel observer for a bioreactor model is provided in Section 5 and the results are compared to the simulation in [1]. Some conclusions are given in Section 6.

We close the introduction with the nomenclature used in this paper:

| | |
|---|---|
| $\mathbb{N}$, $\mathbb{N}_0$ | set of natural numbers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ |
| $\mathbb{R}_{\geq 0}$ | $=[0,\infty)$ |
| $\mathbb{C}_-$ | $=\{\ \lambda\in\mathbb{C}\mid \operatorname{Re}\lambda<0\ \}$ |
| $\mathbb{R}^{n\times m}$ | the set of real $n\times m$ matrices |
| $\mathbf{Gl}_n(\mathbb{R})$ | the group of invertible matrices in $\mathbb{R}^{n\times n}$ |
| $\sigma(A)$ | the spectrum of $A\in\mathbb{R}^{n\times n}$ |
| $\mathscr{L}^\infty_{\mathrm{loc}}(I\to\mathbb{R}^n)$ | the set of locally essentially bounded functions $f:I\to\mathbb{R}^n$, $I\subseteq\mathbb{R}$ an interval |
| $\mathscr{L}^\infty(I\to\mathbb{R}^n)$ | the set of essentially bounded functions $f: I\to\mathbb{R}^n$ with norm |
| $\|f\|_\infty$ | $=\operatorname{ess\,sup}_{t\in I}\|f(t)\|$ |
| $\mathscr{W}^{k,\infty}(I\to\mathbb{R}^n)$ | the set of $k$-times weakly differentiable functions $f:I\to\mathbb{R}^n$ such that $f,\dots,f^{(k)}\in\mathscr{L}^\infty(I\to\mathbb{R}^n)$ |
| $\mathscr{C}^k(I\to\mathbb{R}^n)$ | the set of $k$-times continuously differentiable functions $f:I\to\mathbb{R}^n$ |
| $\mathscr{C}(I\to\mathbb{R}^n)$ | $=\mathscr{C}^0(I\to\mathbb{R}^n)$ |
| $f\|_J$ | restriction of the function $f:I\to\mathbb{R}^n$ to $J\subseteq I$ |

## 2. System Class

In the present paper we consider a large class of nonlinear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t),T(y,\dot y,\dots,y^{(r-1)})(t)\big)\\ &\quad+\Gamma\big(d_\Gamma(t),T_\Gamma(y,\dot y,\dots,y^{(r-1)})(t)\big)u(t),\\ y|_{[-h,0]} &= y^0\in\mathscr{W}^{(r-1),\infty}([-h,0]\to\mathbb{R}^m), \end{aligned} \tag{4}$$

where $h>0$ is the "memory" of the system, $r\in\mathbb{N}$ is the strict relative degree, and

- $d\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^p)$, $d_\Gamma\in\mathscr{W}^{1,\infty}(\mathbb{R}_{\geq 0}\to\mathbb{R}^p)$, $p\in\mathbb{N}$, are disturbances;
- $f\in\mathscr{C}(\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^m)$, $q\in\mathbb{N}$;
- $\Gamma\in\mathscr{C}^1(\mathbb{R}^p\times\mathbb{R}^\ell\to\mathbf{Gl}_m(\mathbb{R}))$ is the high-frequency gain matrix function;
- $T:\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r\to\mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$ is an operator with the following properties:
  a) $T$ maps bounded trajectories to bounded trajectories, i.e., for all bounded $\zeta\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r$ we have $T(\zeta)\in\mathscr{L}^\infty(\mathbb{R}_{\geq 0}\to\mathbb{R}^q)$;
  b) $T$ is causal, i.e., for all $t\geq 0$ and all $\zeta,\xi\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r$:
  $$\zeta|_{[-h,t)}=\xi|_{[-h,t)}\implies T(\zeta)|_{[0,t]}=T(\xi)|_{[0,t]};$$
  c) $T$ is "locally Lipschitz" continuous in the following sense: for all $t\geq 0$ there exist $\tau,\delta,c>0$ such that for all $\zeta,\Delta\zeta\in\mathscr{C}([-h,\infty)\to\mathbb{R}^m)^r$ with $\Delta\zeta|_{[-h,t]}=0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty<\delta$ we have
  $$\Big\|\big(T(\zeta+\Delta\zeta)-T(\zeta)\big)\big|_{[t,t+\tau]}\Big\|_\infty\leq c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

- $T_\Gamma : \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^\ell)$ is an operator with the properties a)–c) and, additionally, $T_\Gamma(\zeta)$ is absolutely continuous for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$. Furthermore, there exists $J \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^{rm} \times \mathbb{R}^\ell \times \mathbb{R}^q \to \mathbb{R}^{m\times m})$ such that for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r$ and almost all $t \geq 0$ we have:

$$\begin{aligned}&\frac{\partial \Gamma(\cdot)^{-1}}{\partial T_\Gamma}\big(d_\Gamma(t), T_\Gamma(\zeta)(t)\big)\,\tfrac{\mathrm{d}}{\mathrm{d}t}T_\Gamma(\zeta)(t)\\ &\qquad = J\big(d_\Gamma(t), \zeta(t), T_\Gamma(\zeta)(t), T(\zeta)(t)\big).\end{aligned} \tag{5}$$

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h,\infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (4), respectively. For fixed $u \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ we call $y \in \mathscr{C}^{r-1}([-h,\omega) \to \mathbb{R}^m)$ a solution of (4) on $[-h,\omega)$, $\omega \in (0,\infty]$, if $y|_{[-h,0]} = y^0$ and $y^{(r-1)}|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (4) for almost all $t \in [0,\omega)$; $y$ is called *maximal*, if it has no right extension that is also a solution. Existence of maximal solutions of (4) for every $y^0 \in \mathscr{W}^{(r-1),\infty}([-h,0] \to \mathbb{R}^m)$ and every $u \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is guaranteed by [7, Thm. 5]; if $y, \dot{y}, \ldots, y^{(r-1)}$ are bounded, then $\omega = \infty$.

In the case of relative degree one, i.e., $r = 1$, systems similar to (4) are well studied, see [6, 7, 9, 14]. For relative degree two systems see [4], and for higher relative degree see [8]. In the aforementioned references it is shown that the class of systems (4) encompasses linear and nonlinear systems with existing strict relative degree and exponentially stable internal dynamics (zero dynamics in the linear case) and the operator $T$ allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, input-to-state stable systems, and combinations thereof. Compared to these works we have added the condition (5) which ensures an input-independent formulation of the observer error dynamics.

In the following we consider some important subclasses of the systems (4).

### 2.1. Minimum-phase finite-dimensional linear systems with strict relative degree

We consider the system (2) with (A1) and (A2) and show that it belongs to the class (4). It is known that systems of this type can be brought into *Byrnes-Isidori form*, see [8]. That is, there exists some $S \in \mathbf{Gl}_n(\mathbb{R})$ such that for $\hat{x}(t) = [x_1(t)^\top, \ldots, x_r(t)^\top, \eta(t)^\top]^\top = Sx(t)$ we have that (2) is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$, $y(t) = \hat{C}\hat{x}(t)$ with

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix},$$
$$\hat{C} = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{6}$$

where $R_1, \ldots, R_r \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times(n-rm)}$, $P \in \mathbb{R}^{(n-rm)\times m}$ and $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$. Further, the minimum-phase property (i) is equivalent to $Q$ being Hurwitz, i.e., $\sigma(Q) \subseteq \mathbb{C}_-$.

The $\mathbb{R}^m$-valued functions $x_1, \ldots, x_r$ satisfy $x_i = y^{(i-1)}$ for $i = 1, \ldots, r$. By further using the variation of constants formula for $\dot{\eta} = Q\eta + Py$, we obtain

$$\begin{aligned} y^{(r)}(t) =& R_1 y(t) + \ldots R_r y^{(r-1)}(t) \\ &+ Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau + CA^{r-1}Bu(t). \end{aligned}$$

This is a system of type (4) with $\Gamma \equiv CA^{r-1}B$ and

$$\begin{aligned} &f(d(t), T(y, \ldots, y^{(r-1)})(t)) = T(y, \ldots, y^{(r-1)})(t) \\ =& R_1 y(t) + \ldots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau. \end{aligned}$$

Note that $T$ is parameterized by $\eta(0) \in \mathbb{R}^{n-rm}$. $T$ is obviously causal and locally Lipschitz, and since $Q$ is Hurwitz, $T$ has the required bounded-input, bounded-output property.

### 2.2. Infinite-dimensional linear systems with exponentially stable zero dynamics and strict relative degree

Consider the system (2), where for some real Hilbert space $X$, the linear operator $A : D(A) \subseteq X \to X$ is the generator of a strongly continuous semigroup, and $B : \mathbb{R}^m \to X$, $C : X \to \mathbb{R}^m$ are linear and bounded. Further assume that the system has the following additional properties:

- The *zero dynamics* of (2) are exponentially stable, that is, there exist $M, \omega > 0$ such that for all solutions of $\dot{x} = Ax + Bu$ with $Cx = 0$ we have $\|x(t)\|_X + \|u(t)\| \leq M\|x(0)\|_X e^{-\omega t}$ for all $t \geq 0$;
- $\operatorname{im} B \subseteq D(A^r)$, $\operatorname{im} C^* \subseteq D((A^*)^r)$, $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

We note that, in the finite-dimensional case, exponential stability of the zero dynamics is equivalent to the system being minimum-phase. It was shown in [10] that this class allows the transformation into a Byrnes-Isidori form (6), where $R_1, \ldots, R_r \in \mathbb{R}^{m\times m}$ and $S : \hat{X} \to \mathbb{R}^m$, $P : \mathbb{R}^m \to \hat{X}$ are bounded linear operators acting on some Hilbert space $\hat{X}$. The operator $Q : D(A) \cap \hat{X} \to \hat{X}$ generates an exponentially stable semigroup $e^{Qt}$ in $\hat{X}$. This system belongs to the class (4) by the same argumentation as for the finite-dimensional case.

### 2.3. Nonlinear systems

Consider the nonlinear input-affine system

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned} \tag{7}$$

with $f \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^n)$, $g \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^{n\times m})$ and $h \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^m)$. We assume that there exists a global diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that the coordinate transformation $[x_1(t)^\top, \ldots, x_r(t)^\top, \eta(t)^\top]^\top = \psi\big(x(t)\big)$ transforms (7) into *input-normalized Byrnes-Isidori form* (see e.g. [11]):

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t),\\
&\vdots\\
\dot{x}_{r-1}(t) &= x_r(t),\\
\dot{x}_r(t) &= g_1(\hat{x}(t),\eta(t)) + g_2(\hat{x}(t),\eta(t))u(t),\\
\dot{\eta}(t) &= g_3(\hat{x}(t),\eta(t)),\\
y(t) &= x_1(t),
\end{aligned}$$

with $\hat{x}(t) = [x_1(t)^\top,\dots,x_r(t)^\top]^\top$, where $g_1 \in \mathscr{C}(\mathbb{R}^n \to \mathbb{R}^m)$, $g_3 \in \mathscr{C}^1(\mathbb{R}^n \to \mathbb{R}^{n-rm})$ and $g_2 \in \mathscr{C}^1(\mathbb{R}^n \to \mathbf{Gl}_m(\mathbb{R}))$; the latter means that the system has (global) strict relative degree $r$. We assume that

$$\frac{\partial g_2(\cdot)^{-1}}{\partial x_r}\, g_2(\cdot) = 0. \tag{8}$$

For fixed $\hat{x} \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{rm})$ and $\eta^0 \in \mathbb{R}^{n-rm}$ we denote the unique maximal solution of the initial value problem

$$\dot{\eta}(t) = g_3(\hat{x}(t),\eta(t)), \quad \eta(0) = \eta^0$$

by $\eta(\cdot;\eta^0,\hat{x}) : [0,\omega) \to \mathbb{R}^{n-rm}$, $\omega \in (0,\infty]$. Similar to [7] we assume that there exists $\kappa \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0})$ and $c > 0$ such that for all $\hat{x} \in \mathscr{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{rm})$ and all $t \in [0,\omega)$ we have

$$\|\eta(t;\eta^0,\hat{x})\| \leq c\left(1 + \operatorname{ess\,sup}_{s\in[0,t]} \kappa(\|\hat{x}(s)\|)\right); \tag{9}$$

this condition in particular implies $\omega = \infty$. Condition (9) on the internal dynamics of (7) resembles Sontag's [16] input-to-state stability, but in fact it is weaker. To show that systems (7) satisfying the above properties belong to the class (4) we set

$$\begin{aligned}
&T(y,\dots,y^{(r-1)})(t)\\
&\quad := \left(y(t)^\top,\dots,y^{(r-1)}(t)^\top, \eta(t;\eta^0,y,\dots,y^{(r-1)})^\top\right)^\top
\end{aligned}$$

and calculate that

$$y^{(r)}(t) = g_1\big(T(y,\dots,y^{(r-1)})(t)\big) + g_2\big(T(y,\dots,y^{(r-1)})(t)\big)\,u(t),$$

which is of the form (4) with $f = g_1$, $\Gamma = g_2$ and $T_\Gamma = T$. The operator $T$ is parameterized by $\eta^0$ and obviously causal and locally Lipschitz. Condition (9) implies the required bounded-input, bounded-output property of $T$, cf. also [7]. To show condition (8) we calculate

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t} T_\Gamma(\hat{x})\\
&= \left(\dot{y}^\top,\dots,(y^{(r)})^\top, \dot{\eta}(t;\eta^0,\hat{x})^\top\right)^\top\\
&= \left(\dot{y}^\top,\dots,(y^{(r-1)})^\top, \big(g_1\big(T(\hat{x})\big) + g_2\big(T(\hat{x})\big)\,u\big)^\top, g_3\big(T(\hat{x})\big)^\top\right)^\top
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\partial g_2(\cdot)^{-1}}{\partial T_\Gamma}\big(T(\hat{x})\big)\,\tfrac{\mathrm{d}}{\mathrm{d}t} T_\Gamma(\hat{x})\\
&\overset{(8)}{=} \frac{\partial g_2(\cdot)^{-1}}{\partial x_1}\big(T(\hat{x})\big)\,\dot{y} + \ldots + \frac{\partial g_2(\cdot)^{-1}}{\partial x_{r-1}}\big(T(\hat{x})\big)\,y^{(r-1)}\\
&\quad + \frac{\partial g_2(\cdot)^{-1}}{\partial x_r}\big(T(\hat{x})\big)\,g_1\big(T(\hat{x})\big) + \frac{\partial g_2(\cdot)^{-1}}{\partial \eta}\big(T(\hat{x})\big)\,g_3\big(T(\hat{x})\big)\\
&=: J\big(\hat{x}, T(\hat{x})\big)
\end{aligned}$$

for some continuous $J : \mathbb{R}^{rm} \times \mathbb{R}^n \to \mathbb{R}^{m\times m}$, which shows (5).

### 2.4. Further classes

In the aforementioned classes of systems which can be transformed into a functional differential equation (4), the operator $T$ is basically the solution operator of a differential equation. We can further consider systems which are of the form (4) with $T$ being of some more involved nature: For instance, $T$ may encompass time delays as well as hysteresis. For a detailed explanation of these classes we refer to [7].

**Remark 2.1.** It is possible to incorporate a more involved dependence on the input and its derivatives in the system class (4) by adding a term

$$g\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), u(t),\dots,u^{(k)}(t)\big) \tag{10}$$

to the right-hand side of the differential equation in (4), where $g \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \to \mathbb{R}^m)$ and $u \in \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is $k$-times weakly differentiable. If $u$ is fix and there exist $\tilde{g} \in \mathscr{C}(\mathbb{R}^{(k+1)m} \to \mathbb{R}^j)$ and $G \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^j \to \mathbb{R}^m)$, where $\tilde{g}$ is bounded, such that

$$\begin{aligned}
&g\big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), u(t),\dots,u^{(k)}(t)\big)\\
&\quad = G\Big(d(t), T(y,\dot{y},\dots,y^{(r-1)})(t), \tilde{g}\big(u(t),\dots,u^{(k)}(t)\big)\Big),
\end{aligned}$$

then $\tilde{g}\big(u(t),\dots,u^{(k)}(t)\big)$ can be rewritten as a bounded "disturbance" $\tilde{d}(t)$ and hence the system is again of type (4). If $u,\dots,u^{(k)}$ are bounded, then this is always possible.

## 3. Observer Design

In this section we consider the funnel observer (1) as a new adaptive high-gain observer which resolves some disadvantages of the adaptive $\lambda$-strip observer proposed in [1] and of the non-adaptive high-gain observer proposed in [18]. The observer in [1] achieves that the error $e_1 = y - z_1$ converges to a $\lambda$-strip $[-\lambda,\lambda]$. However, the gain is monotonically non-decreasing and eventually gets so large that the system gets sensitive to measurement noise. Furthermore, the transient behavior of the error $e_1$ cannot be influenced.

Following the methodology of funnel control, see [7, 5] and the references therein, it is our aim that the funnel observer (1) achieves that the error $e_1 = y - z_1$ evolves within a prescribed performance funnel

$$\mathscr{F}_\varphi := \{\, (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \,\}, \tag{11}$$

which is determined by a function $\varphi$ belonging to

$$\Phi := \left\{ \varphi \in \mathscr{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot\varphi \text{ are bounded,} \\ \varphi(s) > 0 \text{ for all } s > 0, \\ \text{and } \liminf_{s\to\infty} \varphi(s) > 0 \end{array} \right\}.$$

Note that the funnel boundary is given by the reciprocal of $\varphi$, see Figure 2. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e_1(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

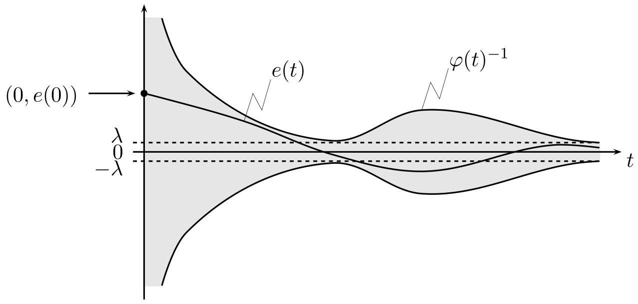


Figure 2: Error evolution in a funnel $\mathscr{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

An important property of the funnel class $\Phi$ is that each performance funnel $\mathscr{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., due to boundedness of $\varphi$ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., when the output signal changes strongly or the system is perturbed by some calibration so that a large observation error would enforce a large observer gain.

The objective is robust estimation of the output $y$ of the system (4) and its derivatives $\dot y, \ldots, y^{(r-1)}$ so that the observation error $e_1 = y - z_1$ evolves within the funnel $\mathscr{F}_\varphi$ and all variables are bounded. To achieve this objective we consider the funnel observer (1) for system (4) with initial conditions

$$z_i(0) = z_i^0 \in \mathbb{R}^m, \quad i = 1, \ldots, r, \tag{12}$$

where $\varphi \in \Phi$, $\tilde\Gamma \in \mathbb{R}^{m\times m}$ and $q_i > 0$, $p_i > 0$ for all $i = 1, \ldots, r$. The functions $z_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $i = 1, \ldots, r$, are the observer states and $k : \mathbb{R}_{\geq 0} \to [1, \infty)$ is the observer gain. The constants $q_i > 0$ are such that the matrix

$$A = \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}$$

is Hurwitz, i.e., $\sigma(A) \subseteq \mathbb{C}_-$. The constants $p_i$ depend on the choice of the $q_i$ in the following way: Let $Q = Q^\top > 0$ and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}, \quad P_{11} \in \mathbb{R},\, P_{12} \in \mathbb{R}^{1\times(r-1)},\, P_{22} \in \mathbb{R}^{(r-1)\times(r-1)}$$

be such that

$$A^\top P + PA + Q = 0, \quad P > 0.$$

The matrix $P$ depends only on the choice of the constants $q_i$ and the matrix $Q$. The constants $p_i$ must then satisfy

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1} P_{12}^\top \end{pmatrix}. \tag{13}$$

This condition guarantees that $P$ defines a quadratic Lyapunov function for the observer error dynamics.

The funnel observer (1) is different in its structure when compared to the high-gain observers in [18, 1], where the gain enters with power $k^i$ into the equation for $\dot z_i$. Furthermore, the constants $q_i$ are not present in [18, 1], but we show that they are important to ensure boundedness of the error dynamics even when $k(t)$ is small.

Although the observer (1) is a nonlinear and time-varying system, it is simple in its structure and its dimension depends only on the relative degree $r$ of the system (4). Apart from the relative degree, no knowledge of the system (1) is required for the construction of the funnel observer (1); it only uses the input signal $u(t)$ and the output signal $y(t)$, see Figure 3. The bounded-input, bounded-output property of the operators $T$ and $T_\Gamma$ in (4) can be exploited for an inherent high-gain property of the system (4) and hence to maintain error evolution within the funnel: by the design of the observer (1), the gain $k(t)$ increases if the norm of the error $\|y(t) - z_1(t)\|$ approaches the funnel boundary $1/\varphi(t)$, and decreases if a high gain is not necessary.

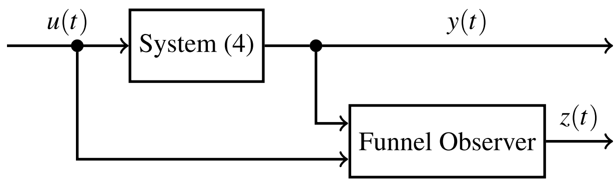


Figure 3: Interconnection of system (4) with the funnel observer (1).

For a sketch of the construction of the funnel observer (1) see also Figure 4.

## 4. Main Result

In this section we prove the main result of the present paper: The funnel observer (1), using $u(t)$ and $y(t)$, provides estimates for all bounded signals $y, \dot y, \ldots, y^{(r-1)}$ of the system (4) such that $y - z_1$ evolves in a prescribed performance funnel $\mathscr{F}_\varphi$ and all signals are bounded; this is true for any disturbances $d$ and $d_\Gamma$, i.e., the observer is robust. We only consider the relevant case of strict relative degree $r \geq 2$.

**Theorem 4.1.** *Consider the system* (4) *with* $r \geq 2$. *Let* $y^0 \in \mathscr{W}^{(r-1),\infty}([-h, 0] \to \mathbb{R}^m)$, $u \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ *and let* $y \in \mathscr{C}^{r-1}([-h, \infty) \to \mathbb{R}^m)$ *be a solution of* (4) *such that* $y, \dot y, \ldots, y^{(r-1)}$ *are bounded. Consider the funnel observer* (1), (12) *with* $\varphi \in \Phi$ *such that*

$$\varphi(0)\|y(0) - z_1^0\| < 1,$$

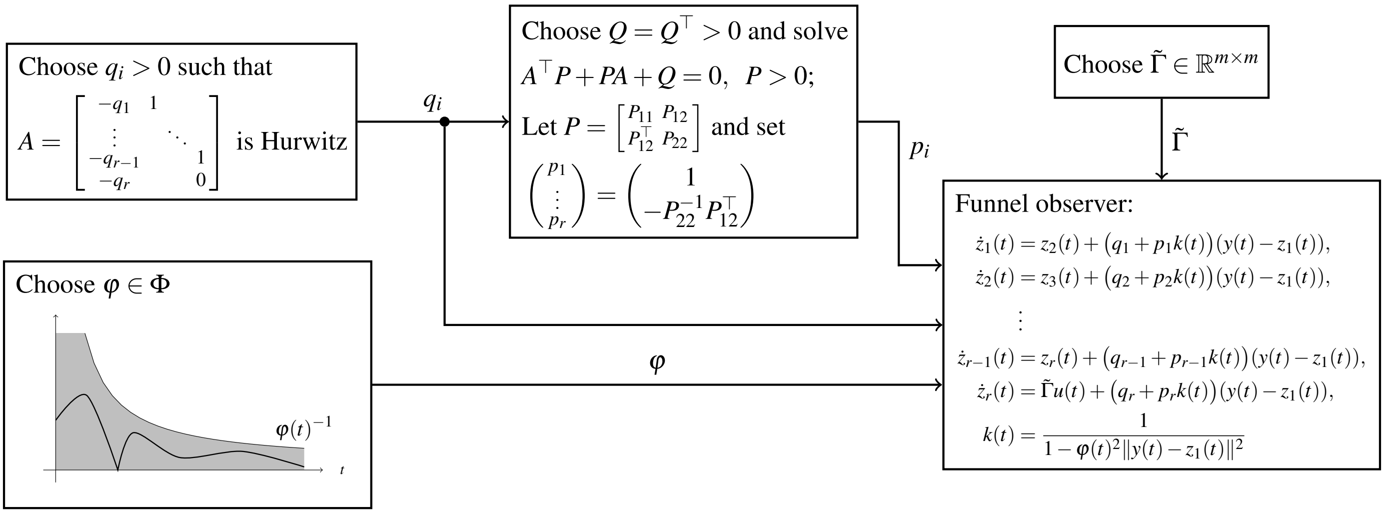


Figure 4: Construction of the funnel observer (1) depending on its design parameters.

$\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ *and* $q_i > 0$, $p_i > 0$ *such that* (13) *is satisfied for corresponding matrices* $A, P, Q$.
*Then* (1), (12) *has an absolutely continuous solution* $z = (z_1, \ldots, z_r) \in \mathscr{L}^\infty\big(\mathbb{R}_{\ge 0} \to (\mathbb{R}^m)^r\big)$ *with* $k \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to [1, \infty))$ *and*

$$\forall\, t > 0: \ \varphi(t)\|y(t) - z_1(t)\| < 1. \tag{14}$$

*Proof.* We proceed in several steps.

*Step 1*: We show existence of a local solution of (1), (12). Set

$$\mathscr{D} := \{\, (t, e_1, \ldots, e_r) \in \mathbb{R}_{\ge 0} \times (\mathbb{R}^m)^r \mid \varphi(t)\|e_1\| < 1 \,\}$$

and

$$\begin{aligned} Y &:= (y, \dot{y}, \ldots, y^{(r-1)}), \\ F(t, Y) &:= \Gamma\big(d_\Gamma(t), T_\Gamma(Y)(t)\big)^{-1} f\big(d(t), T(Y)(t)\big), \\ G(t, Y) &:= \Big( \frac{\partial \Gamma(\cdot)^{-1}}{\partial d_\Gamma} \big(d_\Gamma(t), T_\Gamma(Y)(t)\big)\, \dot{d}_\Gamma(t) \\ &\qquad + J\big(d_\Gamma(t), Y(t), T_\Gamma(Y)(t), T(Y)(t)\big) \Big)\, y^{(r-1)}(t). \end{aligned}$$

Defining

$$\begin{aligned} e_i &:= y^{(i-1)} - z_i, \quad i = 1, \ldots, r-1 \\ e_r &:= \tilde{\Gamma}\Gamma^{-1} y^{(r-1)} - z_r, \end{aligned} \tag{15}$$

and invoking $r \ge 2$ we find

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) - \big(q_1 + p_1 k(t)\big) e_1(t), \\ &\ \ \vdots \\ \dot{e}_{r-2}(t) &= e_{r-1}(t) - \big(q_{r-2} + p_{r-2} k(t)\big) e_1(t), \\ \dot{e}_{r-1}(t) &= e_r(t) - \big(q_{r-1} + p_{r-1} k(t)\big) e_1(t) + (I - \tilde{\Gamma}\Gamma^{-1}) y^{(r-1)}(t), \\ \dot{e}_r(t) &= -\big(q_r + p_r k(t)\big) e_1(t) + \tilde{\Gamma}\big(F(t, Y) + G(t, Y)\big), \\ k(t) &= \frac{1}{1 - \varphi(t)^2 \|e_1(t)\|^2}. \end{aligned} \tag{16}$$

By the existence theorem for ordinary differential equations (see e.g. [19, § 10, Thm. VI]), there exists a maximal absolutely continuous solution $e = (e_1, \ldots, e_r) : [0, \omega) \to (\mathbb{R}^m)^r$, $\omega \in (0, \infty]$, of (16) satisfying the initial conditions

$$\begin{aligned} e_i(0) &= y^{(i-1)}(0) - z_i^0, \quad i = 1, \ldots, r, \\ e_r(0) &= \tilde{\Gamma}\Gamma^{-1} y^{(r-1)}(0) - z_r^0, \end{aligned}$$

and $(t, e(t)) \in \mathscr{D}$ for all $t \in [0, \omega)$. Furthermore, the closure of the graph of $e$, i.e., the set

$$\overline{\text{graph}\, e} := \overline{\{\, (t, e(t)) \mid t \in [0, \omega) \,\}},$$

is not a compact subset of $\mathscr{D}$. Thus, a local solution $(z_1, \ldots, z_r)$ of (1), (12) can be reconstructed.

*Step 2*: We show that $e \in \mathscr{L}^\infty\big([0, \omega) \to (\mathbb{R}^m)^r\big)$. Recalling that the Kronecker product of two matrices $V \in \mathbb{R}^{m \times n}$ and $W \in \mathbb{R}^{p \times q}$ is given by

$$V \otimes W = \begin{bmatrix} v_{11} W & \cdots & v_{1n} W \\ \vdots & & \vdots \\ v_{m1} W & \cdots & v_{mn} W \end{bmatrix} \in \mathbb{R}^{mp \times nq}, \tag{17}$$

let

$$\hat{A} := A \otimes I_m = \begin{bmatrix} -q_1 I_m & I_m & & \\ \vdots & & \ddots & \\ -q_{r-1} I_m & & & I_m \\ -q_r I_m & & & 0 \end{bmatrix} \in \mathbb{R}^{rm \times rm},$$

and, for $P = (p_{ij})_{i,j=1,\ldots,r}$ and $Q = (q_{ij})_{i,j=1,\ldots,r}$,

$$\hat{P} := P \otimes I_m \in \mathbb{R}^{rm \times rm}, \quad \hat{Q} = Q \otimes I_m \in \mathbb{R}^{rm \times rm}.$$

Since the Kronecker product (17) satisfies that, if $m = n$ and $p = q$, then

$$\det(V \otimes W) = (\det V)^p \ (\det W)^m,$$

we obtain that

$$\sigma(\hat{A})=\sigma(A),\quad \sigma(\hat{Q})=\sigma(Q),\quad \sigma(\hat{P})=\sigma(P). \tag{18}$$

Then it follows from $A^\top P+PA+Q=0$ that $\hat{P}=\hat{P}^\top>0$, $\hat{Q}=\hat{Q}^\top>0$ and

$$\hat{A}^\top\hat{P}+\hat{P}\hat{A}+\hat{Q}=0.$$

Since $P_{12}^\top+P_{22}\begin{pmatrix}p_2\\ \vdots\\ p_r\end{pmatrix}=0$ we find

$$\hat{P}\begin{bmatrix}p_1I_m\\ \vdots\\ p_rI_m\end{bmatrix}=\begin{bmatrix}(P_{11}-P_{12}P_{22}^{-1}P_{12}^\top)I_m\\ 0\\ \vdots\\ 0\end{bmatrix},$$

where $P_{11}-P_{12}P_{22}^{-1}P_{12}^\top>0$. Observe that we may write (16) in the form

$$\dot{e}(t)=\hat{A}e(t)-k(t)\begin{bmatrix}p_1I_m\\ \vdots\\ p_rI_m\end{bmatrix}e_1(t)+\begin{pmatrix}0\\ \vdots\\ 0\\ (I-\tilde{\Gamma}\Gamma^{-1})y^{(r-1)}(t)\\ \tilde{\Gamma}\big(F(t,Y)+G(t,Y)\big)\end{pmatrix}.$$

By assumption, $y^{(r-1)}$ is bounded, hence there exists $M_1>0$ such that

$$\forall t\in[0,\omega):\ \|(I-\tilde{\Gamma}\Gamma^{-1})y^{(r-1)}(t)\|\le M_1. \tag{19}$$

Furthermore, by boundedness of $Y$ and the bounded-input, bounded-output property of $T$ and $T_\Gamma$ it follows that $T(Y)$ and $T_\Gamma(Y)$ are bounded, and since $d$ is bounded and $f$ is continuous we have that $f\big(d(\cdot),T(Y)(\cdot)\big)$ is bounded on $[0,\omega)$. As $\Gamma(\cdot)^{-1}$ is continuous and $d_\Gamma$ is bounded we further obtain boundedness of $\Gamma\big(d_\Gamma(\cdot),T_\Gamma(Y)(\cdot)\big)^{-1}$, which yields boundedness of $F(\cdot,Y)$. Similar arguments, using continuity of $\frac{\partial\Gamma(\cdot)^{-1}}{\partial d_\Gamma}$ and $J(\cdot)$ and boundedness of $\dot{d}_\Gamma$ and $y^{(r-1)}$, yield boundedness of $G(\cdot,Y)$, whence we find $M_2>0$ such that

$$\text{for a.a. } t\in[0,\omega):\ \|\tilde{\Gamma}\big(F(t,Y)+G(t,Y)\big)\|\le M_2. \tag{20}$$

Let $M:=\max\{M_1,M_2\}$. We may now calculate that, for almost all $t\in[0,\omega)$,

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}e(t)^\top\hat{P}e(t)\\
&=e(t)^\top\hat{A}^\top\hat{P}e(t)+e(t)^\top\hat{P}\hat{A}e(t)-2k(t)e(t)^\top\hat{P}\begin{bmatrix}p_1I_m\\ \vdots\\ p_rI_m\end{bmatrix}e_1(t)\\
&\quad+2e(t)^\top\hat{P}\begin{pmatrix}0\\ \vdots\\ 0\\ (I-\tilde{\Gamma}\Gamma^{-1})y^{(r-1)}(t)\\ \tilde{\Gamma}\Gamma^{-1}F(t,Y)\end{pmatrix}\\
&\le -e(t)^\top\hat{Q}e(t)-2k(t)(P_{11}-P_{12}P_{22}^{-1}P_{12}^\top)\|e_1(t)\|^2\\
&\quad+2M\|\hat{P}\|\|e(t)\|\\
&\le -\mu e(t)^\top\hat{P}e(t)+2M\|\hat{P}\|\|e(t)\|,
\end{aligned}$$

where $\mu=\lambda_{\min}(\hat{Q})/\lambda_{\max}(\hat{P})$.[1] Now let $\delta\in\big(0,\mu\lambda_{\min}(\hat{P})\big)$ be arbitrary and

$$R=\frac{2M\|\hat{P}\|}{\delta}.$$

Then

$$2M\|\hat{P}\|\|e(t)\|\le\delta\|e(t)\|^2+2M\|\hat{P}\|R \tag{21}$$

provided that $\|e(t)\|\le R$, and if $\|e(t)\|>R$, then

$$2M\|\hat{P}\|\|e(t)\|-\delta\|e(t)\|^2\le\big(2M\|\hat{P}\|-\delta R\big)\|e(t)\|=0,$$

and hence (21) is also true in this case. Therefore,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}e(t)^\top\hat{P}e(t)\le\left(-\mu+\frac{\delta}{\lambda_{\min}(\hat{P})}\right)e(t)^\top\hat{P}e(t)+2M\|\hat{P}\|R$$

for almost all $t\in[0,\omega)$. Gronwall's lemma now implies that, with $\nu=\mu-\frac{\delta}{\lambda_{\min}(\hat{P})}>0$,

$$e(t)^\top\hat{P}e(t)\le e(0)^\top\hat{P}e(0)e^{-\nu t}+\frac{2M\|\hat{P}\|R}{\nu},$$

and hence

$$\|e(t)\|^2\le\frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})}e^{-\nu t}\|e(0)\|^2+\frac{2M\|\hat{P}\|R}{\nu\lambda_{\min}(\hat{P})} \tag{22}$$

for all $t\in[0,\omega)$. Equation (22) in particular implies that $e\in\mathscr{L}^\infty([0,\omega)\to(\mathbb{R}^m)^r)$.

*Step 3*: We show that $k\in\mathscr{L}^\infty([0,\omega)\to\mathbb{R})$. Let $\kappa\in(0,\omega)$ be arbitrary but fixed and $\lambda:=\inf_{t\in(0,\omega)}\varphi(t)^{-1}>0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t\to\infty}\varphi(t)>0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L>0$ of $\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$. By Step 2, $e_2$ is bounded and we may choose $\varepsilon>0$ small enough so that

$$\varepsilon\le\min\left\{\frac{\lambda}{2},\inf_{t\in(0,\kappa]}(\varphi(t)^{-1}-\|e_1(t)\|)\right\}$$

and

$$L\le-\sup_{t\in[0,\omega)}\|e_2(t)\|+\frac{q_1\lambda}{2}+\frac{\lambda^2}{4\varepsilon}; \tag{23}$$

feasibility of this choice is guaranteed by $r\ge2$. We show that

$$\forall t\in(0,\omega):\ \varphi(t)^{-1}-\|e_1(t)\|\ge\varepsilon. \tag{24}$$

By definition of $\varepsilon$ this holds on $(0,\kappa]$. Seeking a contradiction suppose that

$$\exists t_1\in[\kappa,\omega):\ \varphi(t_1)^{-1}-\|e_1(t_1)\|<\varepsilon.$$

[1] Here $\lambda_{\max}(\hat{P})$ denotes the largest eigenvalue of the positive definite matrix $\hat{P}$, and $\lambda_{\min}(\hat{P})$ denotes its smallest eigenvalue.

Then for

$$t_0 := \max\left\{ t \in [\kappa, t_1) \;\middle|\; \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon \right\}$$

we have for all $t \in [t_0, t_1]$ that

$$\varphi(t)^{-1} - \|e_1(t)\| \le \varepsilon,$$

$$\|e_1(t)\| \ge \varphi(t)^{-1} - \varepsilon \ge \lambda - \varepsilon \ge \frac{\lambda}{2}$$

and

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e_1(t)\|^2} \ge \frac{1}{2\varepsilon\varphi(t)} \ge \frac{\lambda}{2\varepsilon}.$$

Now we have, for all $t \in [t_0, t_1]$,

$$\begin{aligned}
&\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = e_1(t)^\top \big(e_2(t) - \big(q_1 + p_1 k(t)\big)e_1(t)\big) \\
&\le -(q_1 + p_1 k(t))\|e_1(t)\|^2 + \left(\sup_{t\in[0,\omega)} \|e_2(t)\|\right)\|e_1(t)\| \\
&\le -\left(\frac{q_1\lambda}{2} + \frac{\lambda^2}{4\varepsilon}\right)\|e_1(t)\| + \left(\sup_{t\in[0,\omega)} \|e_2(t)\|\right)\|e_1(t)\| \\
&\overset{(23)}{\le} -L\|e_1(t)\|.
\end{aligned}$$

Therefore, using

$$\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = \|e_1(t)\| \tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|,$$

and that $\|e_1(t)\| > 0$ for all $t \in [t_0, t_1]$, we find that

$$\begin{aligned}
\|e_1(t_1)\| - \|e_1(t_0)\| &= \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \,\mathrm{d}t \\
&\le -L(t_1 - t_0) \le -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \\
&\le \varphi(t_1)^{-1} - \varphi(t_0)^{-1},
\end{aligned}$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \le \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction. Therefore, (24) holds and this implies boundedness of $k$.

*Step 4*: We show $\omega = \infty$. Assume that $\omega < \infty$. Then, since $e$ and $k$ are bounded by Steps 2 and 3, it follows that $\overline{\text{graph } e}$ is a compact subset of $\mathscr{D}$, a contradiction. Therefore, $\omega = \infty$.

In particular, Steps 3 and 4 imply (14) and this finishes the proof. $\square$

**Remark 4.2.** If the input $u$ is bounded, then the funnel observer works for an even larger system class than (4) and strict relative degree is not required. Consider a system of the form

$$\begin{aligned}
y^{(r)}(t) &= F\big(d_0(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t), u(t), \ldots, u^{(k)}(t)\big) \\
y|_{[-h,0]} &= y^0 \in \mathscr{W}^{(r-1),\infty}([-h,0] \to \mathbb{R}^m),
\end{aligned} \tag{25}$$

where $F \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{(k+1)m} \to \mathbb{R}^m)$, $d_0 \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p)$, $u \in \mathscr{W}^{k,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ and $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^m)^r \to \mathscr{L}^\infty_{\text{loc}}(\mathbb{R}_{\ge 0} \to \mathbb{R}^q)$ is an operator with the properties as discussed in Section 2. It is then possible to reformulate (25) as a system of the form (4). To this end, let $d_1 := \big(u^\top, \ldots, (u^{(k)})^\top\big)^\top$, $d_2 := u$, $d := \big(d_0^\top, d_1^\top, d_2^\top\big)^\top \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{(k+1)m})$ and

$$f : \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^{(k+1)m} \times \mathbb{R}^q,\ (d_0, d_1, d_2, T) \mapsto F(d_0, T, d_1) - d_2.$$

Then (25) is equivalent to

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + u(t),$$

i.e., it is of the form (4) with $\Gamma \equiv I_m$ and in particular condition (5) is always satisfied.

Furthermore, exact knowledge of the number $r$ of derivatives of $y$ involved in (25) (the relative degree in case of (4)) is not required for feasibility of the funnel observer. Only an upper bound $\rho \in \mathbb{N}$ is required, i.e., $r \le \rho$. If $y, \ldots, y^{(\rho)}$ are bounded, then the funnel observer (1) (with $r = \rho$ in (1)) works for (25) in the sense of Theorem 4.1. To see this, the proof of Theorem (4.1) has to be recapitulated with the new observation errors $e_i := y^{(i-1)} - z_i$ for $i = 1, \ldots, \rho$.

In Step 2 of the proof of Theorem 4.1 we chose a constant $\delta$ on which the estimate (22) depends. In the following we show ultimate boundedness of $e$ by choosing $\delta$ in an optimal way.

**Corollary 4.3.** *Use the notation and assumptions from Theorem 4.1, the observation error* (15)*, and the constants* $M_1$ *and* $M_2$ *in the estimates* (19) *and* (20)*, resp. Then, with* $M = \max\{M_1, M_2\}$*, we have*

$$\boxed{\limsup_{t\to\infty} \|e(t)\| \le \frac{4M\,\lambda_{\max}(P)^2}{\lambda_{\min}(Q)\,\lambda_{\min}(P)}.} \tag{26}$$

*Proof.* As shown in the proof of Theorem (4.1) the estimate (22) holds true for all $t \ge 0$ and all $\delta \in \big(0, \mu\lambda_{\min}(\hat{P})\big)$, where $\nu = \mu - \frac{\delta}{\lambda_{\min}(\hat{P})}$ and $\mu = \lambda_{\min}(\hat{Q})/\lambda_{\max}(\hat{P})$. Observe that by (18) we have $\lambda_{\min}(\hat{P}) = \lambda_{\min}(P)$, $\lambda_{\max}(\hat{P}) = \lambda_{\max}(P)$ and $\lambda_{\min}(\hat{Q}) = \lambda_{\min}(Q)$. Furthermore, since $\hat{P}$ is positive definite we have $\|\hat{P}\| = \lambda_{\max}(\hat{P}) = \lambda_{\max}(P)$. By (22) we find that

$$\limsup_{t\to\infty} \|e(t)\| \le \sqrt{\frac{2M\,\lambda_{\max}(P)R}{\nu\,\lambda_{\min}(P)}}.$$

A close look at the $\delta$-dependent expression

$$\frac{R}{\nu} = \frac{2M\,\lambda_{\max}(P)}{\delta\left(\mu - \frac{\delta}{\lambda_{\min}(P)}\right)}$$

reveals that it is minimal for the choice

$$\delta = \frac{\mu\,\lambda_{\min}(P)}{2}.$$

With this choice we obtain

$$\frac{R}{\nu} = \frac{8M\,\lambda_{\max}(P)}{\mu^2\,\lambda_{\min}(P)}$$

from which the assertion (26) follows. $\square$

**Remark 4.4.** We consider two special cases for (4) and the funnel observer (1), and the resulting estimate (26).

(i) $\tilde{\Gamma} = 0$. A careful inspection of the proof of Theorem 4.1 reveals that in this case the condition (5) is superfluous. Furthermore, $M_1$ in (19) can be chosen as $M_1 = \|y^{(r-1)}\|_\infty$ and $M_2 = 0$ in (20). Therefore, we find that $M = \|y^{(r-1)}\|_\infty$ in (26). Note that the choice of $\tilde{\Gamma}$ is independent of (4).

(ii) $\tilde{\Gamma} = \Gamma \in \mathbf{Gl}_m(\mathbb{R})$ and $f = 0$. This means to assume that (4) is of the very special form $y^{(r)}(t) = \Gamma u(t)$ and we have exact knowledge of the invertible matrix $\Gamma$. Then $M_1 = M_2 = 0$ in (19) and (20), resp., and hence $M = 0$ in (26). In particular, this implies that $e(t) \to 0$ and $k(t) \to 1$ for $t \to \infty$.

**Remark 4.5.** If the output of the system (4) is subject to measurement noise, i.e., the funnel observer (1) receives $y + n$ instead of $y$, where $n \in \mathscr{C}^r([-h,\infty) \to \mathbb{R}^m)$ is bounded, then the funnel observer achieves that

$$\forall\, t > 0: \ \varphi(t)\|y(t) + n(t) - z_1(t)\| < 1,$$

which implies

$$\forall\, t > 0: \ \frac{\varphi(t)}{1 + \varphi(t)\|n(t)\|}\|y(t) - z_1(t)\| < 1,$$

i.e., $y - z_1$ evolves in the funnel $\mathscr{F}_\psi$, where $\psi = \frac{\varphi(t)}{1+\varphi(t)\|n(t)\|}$. If an upper bound for $n$ is known, say $\|n(t)\| \le \nu$ for all $t \ge 0$, then

$$\forall\, t > 0: \ \|y(t) - z_1(t)\| < \varphi(t)^{-1} + \nu.$$

Hence, the actual error remains in the wider funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the observer.
If the input of the system (4) is subject to noise before the funnel observer receives it, i.e., $u$ enters system (4) and $u + v$ enters the observer (1), where $v \in \mathscr{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, then the statement of Theorem 4.1 remains the same (the funnel observer still works) and the proof only changes slightly: on the right hand side of the equation for $\dot{e}_r$ in (16) the term $-\tilde{\Gamma}v(t)$ has to be added. Due to boundedness of $v$, the remaining calculations stay the same and only the constant $M_2$ possibly needs to be increased.

**Example 4.6.** We consider two examples for the design parameters $q_i$ and $p_i$ for the funnel observer (1), namely we choose the Hurwitz polynomial

$$t^r + q_1 t^{r-1} + \ldots + q_{r-1} t + q_r = (t+1)^r$$

and $Q = I_r$ for $r = 2$ and $r = 3$.

(i) For $r = 2$, we have $q_1 = 2$ and $q_2 = 1$. The solution of the Lyapunov equation $A^\top P + PA + I_2 = 0$ is given by

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

Then the parameters $p_1, p_2$ in the funnel observer (1) are given by $p_1 = 1$ and $p_2 = \frac{1}{3}$. The funnel observer then reads

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + \big(2 + k(t)\big)\big(y(t) - z_1(t)\big), \\ \dot{z}_2(t) &= \tilde{\Gamma}u(t) + \big(1 + \tfrac{1}{3}k(t)\big)\big(y(t) - z_1(t)\big), \\ k(t) &= \frac{1}{1 - \varphi(t)^2\|y(t) - z_1(t)\|^2}. \end{aligned} \tag{27}$$

The eigenvalues of $P$ are given by $\lambda_1 = 1 + \frac{1}{\sqrt{2}}$ and $\lambda_2 = 1 - \frac{1}{\sqrt{2}}$. Therefore, the estimate (26) becomes

$$\limsup_{t\to\infty} \|e(t)\| \le \frac{4M\left(1 + \frac{1}{\sqrt{2}}\right)^2}{1 - \frac{1}{\sqrt{2}}} = \left(8 + \tfrac{28}{5}\sqrt{2}\right) M$$
$$\approx 15.92M.$$

(ii) For $r = 3$, we have $q_1 = q_2 = 3$ and $q_3 = 1$. The solution of the Lyapunov equation $A^\top P + PA + I_3 = 0$ is given by

$$P = \begin{bmatrix} 1 & -\frac{1}{2} & -1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 4 \end{bmatrix}.$$

Then the parameters $p_1, p_2, p_3$ in the funnel observer (1) are given by $p_1 = 1$, $p_2 = \frac{2}{3}$ and $p_3 = \frac{1}{3}$. The funnel observer then reads

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + \big(3 + k(t)\big)\big(y(t) - z_1(t)\big), \\ \dot{z}_2(t) &= z_3(t) + \big(3 + \tfrac{2}{3}k(t)\big)\big(y(t) - z_1(t)\big), \\ \dot{z}_3(t) &= \tilde{\Gamma}u(t) + \big(1 + \tfrac{1}{3}k(t)\big)\big(y(t) - z_1(t)\big), \\ k(t) &= \frac{1}{1 - \varphi(t)^2\|y(t) - z_1(t)\|^2}. \end{aligned} \tag{28}$$

A numerical computation yields that the eigenvalues of $P$ are given by $\lambda_1 \approx 0.1966$, $\lambda_2 \approx 1.4662$ and $\lambda_3 \approx 4.3372$. Therefore, the estimate (26) becomes

$$\limsup_{t\to\infty} \|e(t)\| \le \frac{4M\lambda_3^2}{\lambda_1} \approx 382.81M.$$

## 5. Simulations

We illustrate the funnel observer by comparing it to the simulations of the $\lambda$-strip observer for a bioreactor model in [1].

We consider the generic model as in [1], cf. also [3]:

$$\begin{aligned}
\dot{m}(t) &= \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t) u(t),\\
\dot{s}(t) &= -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + \big(a_4 - s(t)\big) u(t),\\
y(t) &= m(t),
\end{aligned} \tag{29}$$

where $m(t)$ and $s(t)$ denote the concentrations of the microorganism and the substrate, resp., and $u(t)$ is the substrate inflow rate. All state variables are strictly positive and the parameters are $a_1 = a_2 = a_3 = 1$, $a_4 = 0.1$, $m(0) = 0.075$, and $s(0) = 0.03$. For the simulation we choose the following substrate inflow rate:

$$u(t) = \begin{cases} 0.08, & t \in [0, 30 - \varepsilon] \\ 0.02, & t \in [30 + \varepsilon, 50 - \varepsilon] \\ 0.08, & t \geq 50 + \varepsilon, \end{cases}$$

where $\varepsilon \ll 1$ is some positive constant and on the intervals $(30 - \varepsilon, 30 + \varepsilon)$ and $(50 - \varepsilon, 50 + \varepsilon)$ the function $u$ is chosen such that it is continuously differentiable on $\mathbb{R}_{\geq 0}$. This setup for the bioreactor coincides with that considered in [1], where it is also explained that (29) can be reformulated in the form

$$\ddot{y}(t) = \Phi\big(y(t), \dot{y}(t), u(t), \dot{u}(t)\big).$$

Therefore, invoking Remark 4.2, system (29) belongs to the class (4) with $r = 2$ and $\Gamma \equiv I_m$. Theorem 4.1 thus implies that the funnel observer works for (29). We note that we applied the funnel observer to the original system (29) in the simulation and not to the reformulated system as above.

As design parameters for the funnel observer (1) (see also Figure 4) we choose $\tilde{\Gamma} = 0$, $q_1 = 1$, $q_2 = 0.2$, $p_1 = 1$, $p_2 = \frac{1}{11}$ and

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ t \mapsto \tfrac{1}{2} t e^{-t} + \tfrac{100}{\pi} \arctan t.$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 0.02$ thereafter. Since no knowledge of the initial values for (29) is assumed we set the observer initial values to $z_1^0 = z_2^0 = 0$.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: $10^{-14}$, absolute tolerance: $10^{-10}$). In Figure 5 the simulation of the funnel observer (1) for the bioreactor model (29) with the above stated parameters is depicted. Figure 5a shows the output $m$ and its estimate $m_e$, while Figure 5b show the concentration of the substrate $s$ and its estimate $s_e$. An action of the gain function $k$ in Figure 5c is required only if the error $|m(t) - m_e(t)|$ is close to the funnel boundary $1/\varphi(t)$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises rather sharply by about 0.25. After this initial error correction the gain is nearly equal to 1 for most of the time; only slight corrections are necessary when the input $u(t)$ changes its value at $t = 30$ and $t = 50$. This in particular shows that the gain function $k$ is non-monotone.

Compared to the simulation in [1] we see that the funnel observer achieves better estimation results for $m$ and $s$, while the gain function is much smaller ($k$ is equal to its minimal value 1 most of the time). The main reason for this is that the funnel observer is able to influence the transient behavior of the observation error.

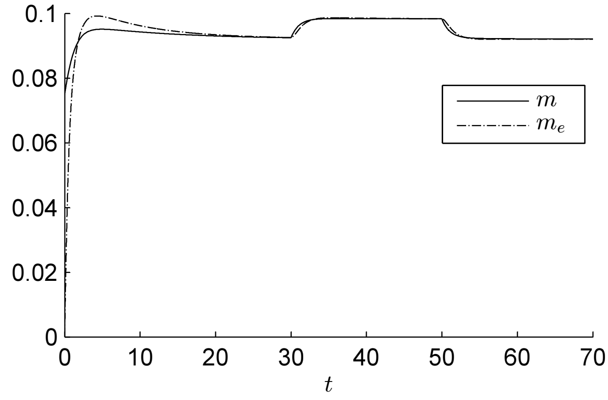


Fig. a: Concentration of microorganism $m$ and its estimate $m_e$

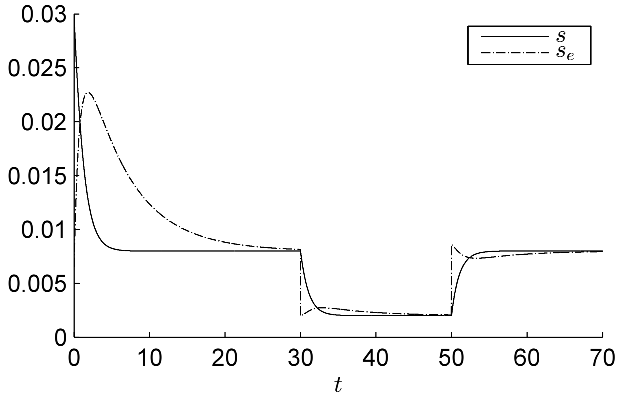


Fig. b: Concentration of substrate $s$ and its estimate $s_e$

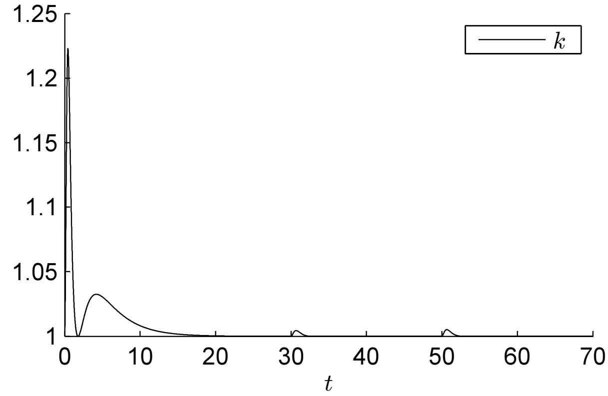


Fig. c: Observer gain $k$

Figure 5: Simulation of the funnel observer (1) for the bioreactor model (29).

## 6. Conclusion

In the present paper we have introduced the funnel observer as a novel and simple adaptive high-gain observer. We showed that the funnel observer is feasible for a large class of nonlinear systems described by functional differential equations which have a known strict relative degree, the internal dynamics map bounded signals to bounded signals, and the operators involved are sufficiently smooth to guarantee local maximal existence of solutions. The proposed adaptation scheme for the observer gain is simple and non-monotone, and we showed that it guarantees prescribed transient behavior of the observation error.

The funnel observer may be used to resolve the problem of higher relative degree in stabilization and tracking problems. If a system has a higher relative degree and derivatives of the output are not available, then a filter or observer is frequently used to obtain approximations of the output derivatives, see the survey [5] and the references therein. As explained there, the concept of funnel control is usually combined with a back-stepping procedure to overcome the higher relative degree, which however complicates the feedback structure. We believe that the funnel observer introduced in the present paper may serve to overcome the obstacle of higher relative degree in funnel control.


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## References


[1] Bullinger, E., Ilchmann, A., Allgöwer, F., 1998. A simple adaptive observer for nonlinear systems, in: Huijberts, H.J.C. (Ed.), Proceedings of the IFAC Symposium Nonlinear Control Systems Design, NOLCOS98, Pergamon, Enschede, The Netherlands. pp. 806–811.

[2] Esfandiari, F., Khalil, H.K., 1987. Observer-based design of uncertain systems: recovering state feedback robustness under matching conditions, in: Proceedings of Allerton Conference, Monticello, IL, pp. 97–106.

[3] Gauthier, J.P., Hammouri, H., Othman, S., 1992. A simple observer for nonlinear systems applications to bioreactors. IEEE Trans. Autom. Control 37, 875 – 880.

[4] Hackl, C.M., Hopfe, N., Ilchmann, A., Mueller, M., Trenn, S., 2013. Funnel control for systems with relative degree two. SIAM J. Control Optim. 51, 965–995.

[5] Ilchmann, A., Ryan, E.P., 2008. High-gain control without identification: a survey. GAMM Mitt. 31, 115–125.

[6] Ilchmann, A., Ryan, E.P., Sangwin, C.J., 2002a. Systems of controlled functional differential equations and adaptive tracking. SIAM J. Control Optim. 40, 1746–1764. URL: `http://link.aip.org/link/?SJC/40/1746/1`, doi:10.1137/S0363012900379704.

[7] Ilchmann, A., Ryan, E.P., Sangwin, C.J., 2002b. Tracking with prescribed transient behaviour. ESAIM: Control, Optimisation and Calculus of Variations 7, 471–493.

[8] Ilchmann, A., Ryan, E.P., Townsend, P., 2007. Tracking with prescribed transient behavior for nonlinear systems of known relative degree. SIAM J. Control Optim. 46, 210–230. URL: `http://link.aip.org/link/?SJC/46/210/1`, doi:10.1137/050641946.

[9] Ilchmann, A., Ryan, E.P., Trenn, S., 2005. Tracking control: Performance funnels and prescribed transient behaviour. Syst. Control Lett. 54, 655–670.

[10] Ilchmann, A., Selig, T., Trunk, C., 2013. The Byrnes-Isidori form for infinite-dimensional systems. Submitted to *SIAM J. on Control and Opt.*; Preprint 10-07 available online, Institute for Mathematics, Ilmenau University of Technology.

[11] Isidori, A., 1995. Nonlinear Control Systems. Communications and Control Engineering Series. 3rd ed., Springer-Verlag, Berlin.

[12] Khalil, H.K., Praly, L., 2014. High-gain observers in nonlinear feedback control. Int. J. Robust & Nonlinear Control 24, 993–1015.

[13] Khalil, H.K., Saberi, A., 2007. Adaptive stabilization of a class on nonlinear systems using high-gain feedback. IEEE Trans. Autom. Control 32, 1031–1035.

[14] Ryan, E.P., Sangwin, C.J., 2001. Controlled functional differential equations and adaptive stabilization. Int. J. Control 74, 77–90.

[15] Saberi, A., Sannuti, P., 1990. Observer design for loop transfer recovery and for uncertain dynamical systems. IEEE Trans. Autom. Control 35, 878–897.

[16] Sontag, E.D., 1989. Smooth stabilization implies coprime factorization. IEEE Trans. Autom. Control 34, 435–443.

[17] Tornambè, A., 1988. Use of asymptotic observers having high gains in the state and parameter estimation, in: Proc. 27th IEEE Conf. Decis. Control, Austin, Texas, pp. 1791–1794.

[18] Tornambè, A., 1992. High-gain observers for non-linear systems. Int. J. Systems Sci. 23, 1475–1489.

[19] Walter, W., 1998. Ordinary Differential Equations. Springer-Verlag, New York.