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Abstract

In this paper we revisit the Kalman-Yakubovich-Popov lemma for differential-algebraic control systems. This lemma relates the positive semi-definiteness of the Popov function on the imaginary axis to the solvability of a linear matrix inequality on a certain subspace. Further emphasis is placed on the Lur'e equation, whose solution set consists, loosely speaking, of the rank-minimizing solutions of the Kalman-Yakubovich-Popov inequality. We show that there is a correspondence from the solution set of the Lur'e equation to deflating subspaces of certain even matrix pencils. Finally, we show that under certain conditions the Lur'e equation admits stabilizing, anti-stabilizing, and extremal solutions. We note that, for our results, we neither assume impulse controllability nor we make any assumptions on the index of the system.

Keywords: differential-algebraic equations, Kalman-Yakubovich-Popov lemma, even matrix pencils, Lur'e equations, algebraic Riccati equations

1 Introduction

In this work we consider differential-algebraic control systems (or descriptor systems) of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where E, $A \in \mathbb{K}^{n\times n}$ such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular (see Def. 2.1 (a)) and $B \in \mathbb{K}^{n\times m}$ (for the notation of this article we refer to the end of this introductory section). The set of these systems is denoted by $\Sigma_{n,m}(\mathbb{K})$ and we write $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. The function $u : \mathbb{R} \to \mathbb{K}^m$ is called *input* of the system; we call $x(t) \in \mathbb{K}^n$ the *(generalized) state* of $[E, A, B]$ at time $t \in \mathbb{R}$. The set of solution trajectories $(x, u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ induces the *behavior* of (1.1):

$$\mathfrak{B}_{[E,A,B]} := \big\{(x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m) : E\dot{x} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \\ \text{and } (x, u) \text{ solves (1.1) for almost all } t \in \mathbb{R}\big\}.$$

The main algebraic concept for our considerations is the *Popov function*, which is defined by

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}(s)^{m\times m},$$

where $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ are given matrices. Note that $\Phi(\mathrm{i}\omega)$ is Hermitian for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. In particular, we are going to study algebraic conditions for the pointwise

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positive semi-definiteness of $\Phi(\mathrm{i}\,\cdot\,) : \{\omega \in \mathbb{R} : \det(\mathrm{i}\omega E - A) \neq 0\} \to \mathbb{C}^{m\times m}$. This property is strongly related to the feasibility of the linear-quadratic optimal control problem in which the cost functional is formed by the matrix $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$, see, e.g., [44].

In the case of ordinary differential equations (that is, $E = I_n$), the pointwise positive semi-definiteness of $\Phi(\mathrm{i}\cdot)$ can be assessed by the famous Kalman-Yakubovich-Popov lemma, see, e.g., [1,21,36,37,47] and references therein. More precisely, under certain assumptions related to controllability, this property is equivalent to the solvability of the Kalman-Yakubovich-Popov (KYP) inequality, namely there exists a $P \in \mathbb{K}^{n\times n}$ such that

$$\begin{bmatrix} A^*P + PA + Q & PB + S \\ B^*P + S^* & R \end{bmatrix} \geq 0, \quad P = P^*. \tag{1.2}$$

There are several attempts to generalize this lemma to differential-algebraic equations: For instance, in [32], the case where $sE - A$ is regular and of index at most one has been treated. In [8, 9, 46] the KYP inequality has been considered for the even more general class of *linear time-invariant behaviors*. In these articles, behavioral controllability has been assumed and essentially used in the proofs. Besides this additional assumption, our approach is completely different in the sense that our considerations are based on the use another type of mathematical tools: While the behavioral approach in [8, 9, 46] is based on tools of polynomial algebra (in particular, the so-called “Smith form”) this article uses the theory of “matrix pencils” [16]. The latter is a subarea of linear algebra.

Other authors treat special cases of the KYP lemma for differential-algebraic systems: For instance, the *positive real lemma* has been considered in [15] (this corresponds to special choices of Q, S, and R). The latter article however contains restricting and artificial assumptions on the system to prove the main result. These assumptions are dropped in [11] by considering a linear matrix inequality related to (1.2) on a certain subspace. We employ a similar idea to present a new, more general version of the KYP lemma for differential-algebraic systems. To this end, we introduce what we mean be equality and positive semi-definiteness on some subspace.

Definition 1.1. Let $\mathcal{V} \subset \mathbb{K}^n$ be a subspace and M, $N \in \mathbb{K}^{n\times n}$ be Hermitian. Then we write

$$\begin{aligned} M =_{\mathcal{V}} N \quad &:\Longleftrightarrow \quad x^*Mx = x^*Nx \quad \forall\, x \in \mathcal{V}, \\ M \geq_{\mathcal{V}} N \quad &:\Longleftrightarrow \quad x^*Mx \geq x^*Nx \quad \forall\, x \in \mathcal{V}. \end{aligned}$$

We will relate pointwise positive semi-definiteness of $\Phi(s)$ on the imaginary axis to the solvability of the *KYP inequality*, see Section 4. By the latter, we mean the existence of some $P \in \mathbb{K}^{n\times n}$, such that

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad P = P^*, \tag{1.3}$$

where $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$ is the largest subspace such that for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ and almost all $t \in \mathbb{R}$ it holds $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) \in \mathcal{V}_{\text{sys}}$. Note that, if (1.1) is an ordinary differential equation (ODE), we have $\mathcal{V}_{\text{sys}} = \mathbb{K}^{n+m}$. We will present an algebraic characterization of $\mathcal{V}_{\text{sys}}$ for general differential-algebraic systems in Section 3.

To study the solution structure of the KYP inequality, we introduce a new type of matrix equation, namely, the *Lur'e equation* for differential-algebraic systems (see Section 5)

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{1.4}$$

that has be solved for a triple $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ for some $q \in \mathbb{N}_0$, such that

$$\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q.$$

It can be seen that Lur'e equation (1.3) defines special solutions of the KYP inequality (1.2). These are *rank-minimizing* in a certain sense. We show that, under some conditions related to controllability of (1.1), there exist *stabilizing* and *anti-stabilizing* solutions, which means that additionally it holds

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall\, \lambda \in \mathbb{C}_+ \quad (\text{or } \forall\, \lambda \in \mathbb{C}_-, \text{ respectively}).$$

We prove that the stabilizing and anti-stabilizing solutions are distinguished in a way that these define *extremal solutions* of the KYP inequality, where "extremal" has to be understood in terms of definiteness of E^*XE. These results are well-known in the ODE case. If E is the identity, then (1.4) reduces to the Lur'e equation treated in [2, 38].

We further show that Lur'e equations (1.4) are intimately connected to certain deflating subspaces of the associated *even matrix pencil*

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -s\Pi E + A & B \\ sE^*\Pi^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \in \mathbb{K}[s]^{2n+m\times 2n+m},$$

where Π is an index-reducing projector, see Section 6 for details. Thereby we generalize the results from [38], where the ODE case has been treated. The proofs of all our results are simple: They are based on a feedback transformation which allows to use the well-known results for the ODE case.

We prove that the solutions of the Lur'e equations (1.4) give rise to the solution of the linear-quadratic optimal control problem. That is, we consider the minimization of the cost functional

$$\mathcal{J}(x,u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau$$

subject to the differential-algebraic equation (1.1) with $Ex_0 = Ex(0)$ and $\lim_{t\to\infty} Ex(t) = 0$.

Note that, on the other hand, the Lur'e equation generalizes the famous *algebraic Riccati equation (ARE)* [30, 44]

$$A^*X + XA - (XB + S)R^{-1}(B^*X + S^*) + Q = 0, \quad X = X^*. \tag{1.5}$$

Namely, if R is invertible (this corresponds to a full weighting of the input), then K and L can be eliminated from the Lur'e equation, which results in (1.5).

The famousness of the ARE has led to various investigations on generalizations of AREs to the differential-algebraic case: In [3, 31, 33], *generalized AREs* of the form

$$A^*XE + E^*XA - (E^*XB + S)R^{-1}(B^*XE + S^*) + Q = 0, \quad X = X^*, \tag{1.6}$$

is studied, whereas in [22–24, 24, 28, 29] *generalized AREs* of the form

$$A^*X + X^*A - (X^*B + S)R^{-1}(B^*X + S^*) + Q = 0, \quad E^*X = X^*E, \tag{1.7}$$

are investigated. We will present the assumptions for these approaches in more detail in Section 8.2. Note that the Riccati equation approach obviously presumes the invertibility of the input weight R.

We will see that positive semi-definiteness of the Popov function on the imaginary axis (except for the poles) is a necessary condition for solvability of the generalized AREs (1.6) and (1.7) as well as for the Lur'e equation (1.4) and the KYP inequality (1.3). The sufficient conditions for solvability of the Lur'e equation will however turn out to be by far weaker than those for generalized AREs. Additional criteria for sufficiency of the solvability will only be related to behavioral stabilizability and behavioral controllability of the system (1.1). These conditions are equivalent to those made in [14, 40].

Further note that, in the ODE case, invertibility of R is equivalent to the regularity of the corresponding optimal control problem [13]. This is no longer true in the differential-algebraic case [17, 18]. The invertibility assumption on R thus becomes an unnecessary artificial assumption for differential-algebraic systems.

These assertions lead us to one of the main conclusion of this work:

For differential-algebraic systems, the generalization of the Kalman-Yakubovich-Popov inequality and Lur'e equation is by far more profitable than the generalization of the algebraic Riccati equation!

Nomenclature

We use the standard notations i, $\overline{\lambda}$, A^*, A^{-*}, I_n, $0_{m\times n}$ for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix and its inverse, the identity matrix of size $n \times n$ and the zero matrix of size $m \times n$ (subscripts are omitted, if clear from context). Further, the following sets are used throughout this article:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, resp.
$\mathbb{K}$	either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers
$\mathbb{C}_+$, $\mathbb{C}_-$	the open sets of complex numbers with positive and negative real part, resp.
$\mathrm{i}\mathbb{R}$	the imaginary axis
$\overline{S}$	closure of the set S
$\mathbb{K}[s]$, $\mathbb{K}(s)$	the ring of polynomials and the field of rational functions with coefficients in $\mathbb{K}$, resp.
$\mathcal{R}^{m\times n}$	the set of $m \times n$ matrices with entries in a ring $\mathcal{R}$
$\mathrm{Gl}_n(\mathbb{K})$	the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$
$\mathcal{L}^2(\mathcal{I}, \mathbb{K}^n)$	the set of measurable and square integrable functions $f : \mathcal{I} \to \mathbb{K}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{K}^n)$	the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathbb{K}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$

Moreover, the blockdiagonal matrix composed of $A_i \in \mathbb{K}^{m_i\times n_i}$ with m_i, $n_i \in \mathbb{N}_0$ for $i = 1, \ldots, k$ is denoted by

$$\mathrm{diag}(A_1, \ldots, A_k) \in \mathbb{K}^{m\times n},$$

where $m = m_1 + \ldots + m_k$, $n = n_1 + \ldots + n_k$. Finally, the rank of $P(s) \in \mathbb{K}(s)^{m\times n}$ over the field $\mathbb{K}(s)$ (often called the normal rank of $P(s)$) is denoted by $\mathrm{rank}_{\mathbb{K}(s)} P(s)$.

2 Control and Matrix Theoretic Preliminaries

2.1 Matrix Pencils

In this subsection we briefly consider matrix pencils, i.e., first order matrix polynomials.

Definition 2.1 (Regularity, generalized eigenvalues)**.** Let a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ be given.

(a) The pencil $sE - A$ is called *regular*, if $m = n$ and $\mathrm{rank}_{\mathbb{K}(s)}(sE - A) = n$. Otherwise it is called *singular*.

(b) A complex number $\lambda \in \mathbb{C}$ is called *(generalized) eigenvalue* of the pencil $sE - A$, if

$$\mathrm{rank}(\lambda E - A) < \mathrm{rank}_{\mathbb{K}(s)}(sE - A).$$

For a regular matrix pencil $sE - A \in \mathbb{K}[s]^{n\times n}$, there exist U_l, $U_\mathrm{r} \in \mathrm{Gl}_n(\mathbb{K})$ such that $U_\mathrm{l}(sE - A)U_\mathrm{r}$ is in *quasi-Weierstrass form* [5], that is

$$U_\mathrm{l}(sE - A)U_\mathrm{r} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 \\ 0 & sE_{22} - I_{n_2} \end{bmatrix},$$

for some $A_{11} \in \mathbb{K}^{n_1\times n_1}$, and a nilpotent matrix $E_{22} \in \mathbb{K}^{n_2\times n_2}$. The nilpotency index of E_{22} (that is, $\nu \in \mathbb{N}_0$ with $E_{22}^{\nu-1} \neq 0$ and $E_{22}^{\nu} = 0$) is called *(Kronecker) index* of $sE - A$.

2.2 Controllability and Stabilizability

In this subsection we present some concepts for controllability and stabilizability of differential-algebraic systems (1.1). We use the definitions from the overview article [6]. To this end, we introduce the vector space of *consistent initial differential variables* of $[E, A, B]$, which is given by

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x_0 \in \mathbb{K}^n \;\middle|\; \exists\, (x, u) \in \mathfrak{B}_{[E,A,B]} \text{ such that } Ex(0) = Ex_0 \right\}. \tag{2.1}$$

A geometric characterization of the vector space of consistent initial differential variables in terms of invariant subspaces can be found in [4, Def. 3.1.5].

Definition 2.2 (Impulse controllability, behavioral (anti-)stabilizability, behavioral controllability)**.** The system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the space of consistent initial differential variables $\mathcal{V}_{\text{diff}}$ is called

$$
\begin{aligned}
\textit{impulse controllable} \quad &:\Longleftrightarrow \quad \forall\, x_0 \in \mathbb{K}^n\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \quad \Longleftrightarrow \quad \mathcal{V}_{\text{diff}} = \mathbb{K}^n;\\
\textit{behavioral stabilizable} \quad &:\Longleftrightarrow \quad \forall\, (x,u) \in \mathfrak{B}_{[E,A,B]}\ \exists\, (\widetilde{x},\widetilde{u}) \in \mathfrak{B}_{[E,A,B]} : (x,u)|_{(-\infty,0)} = (\widetilde{x},\widetilde{u})|_{(-\infty,0)} \text{ and}\\
&\qquad\quad \lim_{t\to\infty} \operatorname{ess\,sup}_{\tau>t} \|(\widetilde{x}(\tau),\widetilde{u}(\tau))\| = 0;\\
\textit{behavioral anti-stabilizable} \quad &:\Longleftrightarrow \quad \forall\, (x,u) \in \mathfrak{B}_{[E,A,B]}\ \exists\, (\widetilde{x},\widetilde{u}) \in \mathfrak{B}_{[E,A,B]} : (x,u)|_{(0,\infty)} = (\widetilde{x},\widetilde{u})|_{(0,\infty)} \text{ and}\\
&\qquad\quad \lim_{t\to-\infty} \operatorname{ess\,sup}_{\tau<t} \|(\widetilde{x}(\tau),\widetilde{u}(\tau))\| = 0;\\
\textit{behavioral controllable} \quad &:\Longleftrightarrow \quad \forall\, (x_1,u_1),\ (x_2,u_2) \in \mathfrak{B}_{[E,A,B]}\ \exists\, T>0,\ (x,u) \in \mathfrak{B}_{[E,A,B]} \text{ with}\\
&\qquad\quad (x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)) & \text{for } t<0,\\ (x_2(t),u_2(t)) & \text{for } t>T. \end{cases}
\end{aligned}
$$

Well known characterizations of these concepts are the following.

Proposition 2.3 (Algebraic controllability and stabilizability characterizations)**.** *Let the system* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *be given with and, for* $r = \operatorname{rank} E$, *assume that* $S_\infty \in \mathbb{K}^{n\times n-r}$ *is matrix with* $\operatorname{im} S_\infty = \ker E$. *Then* $[E, A, B]$ *is*

$$
\begin{aligned}
&\text{(a)} & \textit{impulse controllable} \quad &\Longleftrightarrow \quad \operatorname{rank}\begin{bmatrix} E & AS_\infty & B \end{bmatrix} = n,\\
&\text{(b)} & \textit{behavioral stabilizable} \quad &\Longleftrightarrow \quad \forall\, \lambda \in \overline{\mathbb{C}_+}:\ \operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n,\\
&\text{(c)} & \textit{behavioral anti-stabilizable} \quad &\Longleftrightarrow \quad \forall\, \lambda \in \overline{\mathbb{C}_-}:\ \operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n,\\
&\text{(d)} & \textit{behavioral controllable} \quad &\Longleftrightarrow \quad \forall\, \lambda \in \mathbb{C}:\ \operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n.
\end{aligned}
$$

Proof. Assertions (a), (b), and (d) have been proven in [4]. Statement (c) follows from the simple fact that $[E, A, B]$ is behavioral anti-stabilizable if and only if $[-E, A, B]$ is behavioral stabilizable. ☐

Motivated by the previous result, we introduce the notion of an uncontrollable mode.

Definition 2.4 (Uncontrollable mode)**.** The number $\lambda \in \mathbb{C}$ is called an *uncontrollable mode* of the system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$, if $\operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} < n$.

We further define a concept that is defined by a purely linear algebraic condition and does not have any evident interpretation in terms of the behavior $\mathfrak{B}_{[E,A,B]}$. It generalizes the concept of sign-controllability for systems governed by ordinary differential equations [14, 39, 40].

Definition 2.5 (Behavioral sign-controllability)**.** The system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is called *behavioral sign-controllable* if it holds

$$\operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix} = n \quad \text{or} \quad \operatorname{rank}\begin{bmatrix} -\overline{\lambda} E - A & B \end{bmatrix} = n \quad \forall\, \lambda \in \mathbb{C}.$$

Remark 2.6*.* For ODE systems, behavioral (sign-)controllability and behavioral (anti-)stabilizability reduce to the respective concepts of (sign-)controllability and (anti-)stabilizability in the sense of [14, 38, 40, 42].

2.3 Feedback equivalence

Here we introduce equivalence relations on $\Sigma_{n,m}(\mathbb{K})$, which are the basis for many of the presented proofs.

Definition 2.7 (Feedback equivalence)**.**
Two systems $[E_i, A_i, B_i] \in \Sigma_{n,m}(\mathbb{K})$, $i = 1, 2$, are called *feedback equivalent*, if

$$\exists\, W, T \in \mathrm{Gl}_n(\mathbb{K}) \ \text{ and } F \in \mathbb{K}^{m\times n} : W \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T & 0 \\ -FT & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix}. \tag{2.2}$$

Remark 2.8. The set of uncontrollable modes, impulse controllability, behavioral (anti-)stabilizability and behavioral (sign-)-controllability are invariant under feedback equivalence.

The following feedback equivalence form will we very useful for our proofs.

Proposition 2.9 (Feedback equivalence form)**.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. Then there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ such that*

$$W \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ -FT & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 & B_1 \\ 0 & -I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}, \tag{2.3}$$

where $E_{33} \in \mathbb{K}^{n_3\times n_3}$ is nilpotent. Furthermore, the following statements hold true:

(a) $(x, u) \in \mathfrak{B}_{[E,A,B]} \iff (x_1, u - Fx) \in \mathfrak{B}_{[I_{n_1},A_{11},B_1]}$, *where* $x = T \left(\begin{smallmatrix} x_1 \\ B_2(u-Fx) \\ 0 \end{smallmatrix} \right)$.

(b) The space of consistent initial differential variables fulfills

$$\mathcal{V}_{\mathrm{diff}} = T \left(\mathbb{K}^{n_1+n_2} \times \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix} \right).$$

(c) It holds

$$\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n_2 + n_3 + \operatorname{rank} \begin{bmatrix} \lambda I_{n_1} - A_{11} & B_1 \end{bmatrix} \quad \forall\, \lambda \in \mathbb{C}.$$

In particular, $\lambda \in \mathbb{C}$ is an uncontrollable mode of $[E, A, B]$ if and only if λ is an uncontrollable mode of $[I_{n_1}, A_{11}, B_1]$.

(d) If $[E, A, B]$ is impulse controllable, then W, T, F can be chosen such that $n_3 = 0$.

Proof. The existence of a form

$$\begin{bmatrix} s\widetilde{E} - \widetilde{A} & \widetilde{B} \end{bmatrix} := W_1 \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ -F_1T_1 & I_m \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & sE_{13} - A_{13} & B_1 \\ 0 & -I_{n_2} & sE_{23} - A_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix},$$

is subject of [20, Prop. 2.12]. By using [7, Cor. 2.3], there exist $W_2, T_2 \in \mathrm{Gl}_n(\mathbb{K})$ such that $W_2W_1B = W_1B$ and matrices E_{13} A_{13} are eliminated in $W_2W_1(sE - (A + BF))T_1T_2$. The matrix A_{23} can be further eliminated by a transformation $W_2W_1(sE - (A + BF))T_1T_2$ with $T_3 \in \mathrm{Gl}_n(\mathbb{K})$. Consequently, the form (2.3) is achieved for $W = W_2W_1$, $T = T_1T_2T_3$, and $F = F_1$.

The statements (a), (b), (c), and (d) then follow from [20, Prop. 2.12]. □

Remark 2.10 (Feedback equivalence form). Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with feedback equivalence form (2.3) be given. We can then conclude from Proposition 2.9 (c), Proposition 2.3 and Remark 2.6 that $[E, A, B]$ is behavioral stabilizable (anti-stabilizable, sign-controllable) if and only if the ODE system $[I_{n_1}, A_{11}, B_1]$ is stabilizable (anti-stabilizable, sign-controllable).

3 The system space

The following space will play a crucial role in this article:

Definition 3.1 (System space)**.** The *system space* of $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is the largest subspace $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$, such that for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ it holds

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \text{for almost all } t \in \mathbb{R}.$$

Next we characterize the system space algebraically. First we present a result about the relation between the system spaces of two feedback equivalent systems.

Lemma 3.2. *Assume that $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$ is given. Moreover, let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be such that $sE-(A+BF)$ is regular, and let $\mathcal{V}_{\mathrm{sys},F}$ be the system space of $[WET, W(A+BF)T, WB] \in \Sigma_{n,m}(\mathbb{K})$. Then*

$$\mathcal{V}_{\mathrm{sys}} = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \cdot \mathcal{V}_{\mathrm{sys},F}. \tag{3.1}$$

Proof. The result follows from the findings in [6, p. 16], where it is stated that $(x, u) \in \mathfrak{B}_{[WET,W(A+BF)T,WB]}$ if and only if $(Tx, FTx + u) \in \mathfrak{B}_{[E,A,B]}$. □

Proposition 3.3. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$ be given. Let $(\mathcal{V}_k)$ be a sequence of subspaces with*

$$\begin{aligned} \mathcal{V}_0 &= \mathbb{K}^{n+m}, \\ \mathcal{V}_{k+1} &= \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} \;\middle|\; Ax + Bu \in \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_k \right\}. \end{aligned}$$

These spaces fulfill $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ for all $k \in \mathbb{N}_0$. Moreover, there exists some $k_0 \in \mathbb{N}$ such that $\mathcal{V}_{k_0} = \mathcal{V}_{k_0+i}$ for all $i \in \mathbb{N}_0$. Then the system space fulfills $\mathcal{V}_{\mathrm{sys}} = \mathcal{V}_{k_0}$.

Proof. It can be easily verified that, under feedback transformation (2.3), the spaces $\mathcal{V}_i$ and $\mathcal{V}_{i,F}$ (the latter is the above chain applied to the system $[WET, W(A + BF)T, WB]$) are related by

$$\mathcal{V}_k = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \cdot \mathcal{V}_{k,F} \quad \forall k \in \mathbb{N}_0.$$

Therefore, it suffices to verify this fact for the case where $\begin{bmatrix} sE - A & B \end{bmatrix}$ is in feedback equivalence form (2.3): Using Proposition 2.9 (b), it follows from the nilpotency of E_{33} that

$$\mathcal{V}_{\mathrm{sys}} = \left\{ \begin{pmatrix} x_1 \\ -B_2 u \\ 0_{n_3\times 1} \\ u \end{pmatrix} \;\middle|\; x_1 \in \mathbb{K}^{n_1},\, u \in \mathbb{K}^m \right\}. \tag{3.2}$$

On the other hand, by determining the chain $(\mathcal{V}_i)$ for the system in feedback equivalence form, we can see that

$$\mathcal{V}_i = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ u \end{pmatrix} \;\middle|\; x_1 \in \mathbb{K}^{n_1},\, x_2 \in \mathbb{K}^{n_2},\, x_3 \in \mathbb{K}^{n_3},\, u \in \mathbb{K}^m,\, \begin{pmatrix} x_2 + B_2 u \\ x_3 \end{pmatrix} \in \operatorname{im} \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix}^i \right\}.$$

The nilpotency of E_{33} then implies the desired result. □

Remark 3.4. We can immediately infer from Proposition 2.9 (d) that for an impulse controllable system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ it holds

$$\mathcal{V} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{K}^{n+m} : Ax + Bu \in \operatorname{im} E \right\}. \tag{3.3}$$

We finally prove an auxiliary result on the system space.

Lemma 3.5. *For $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$ it holds*

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \subset \mathcal{V}_{\mathrm{sys}} \quad \forall \lambda \in \mathbb{C} : \det(\lambda E - A) \neq 0. \tag{3.4}$$

Proof. By Proposition 3.3, it suffices to prove that

$$\operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1} B \\ I_m \end{bmatrix} \subset \mathcal{V}_k \quad \forall k \in \mathbb{N}_0,\, \lambda \in \mathbb{C} : \det(\lambda E - A) \neq 0, \tag{3.5}$$

which is done by induction:
For $k = 0$, the statement is trivial. As induction hypothesis, we assume that the statement holds true for $k-1$. By using the identity

$$A(sE - A)^{-1}B = (sE - (sE - A))(sE - A)^{-1}B = sE(sE - A)^{-1}B - B,$$

we obtain

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} = s \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix}. \tag{3.6}$$

Assume that $\lambda \in \mathbb{C}$ with $\det(\lambda E - A) \neq 0$. The previous identity together with the induction hypothesis gives

$$\begin{bmatrix} A & B \end{bmatrix} \operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} = \begin{bmatrix} E & 0 \end{bmatrix} \operatorname{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \subset \begin{bmatrix} E & 0 \end{bmatrix} \mathcal{V}_{k-1}.$$

The definition of $\mathcal{V}_k$ then yields (3.5). □

4 The Kalman-Yakubovich-Popov inequality

In this section we present a differential-algebraic version of the Kalman-Yakubovich-Popov (KYP) lemma. This means that we equivalently characterize the positive semi-definiteness of the Popov function

$$\Phi(s) = \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}(s)^{m \times m}, \tag{4.1}$$

on the imaginary axis by the solvability of a linear matrix inequality. By the latter, we mean the existence of some $P \in \mathbb{K}^{n \times n}$ with

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad P = P^*. \tag{4.2}$$

The following theorem is the main result of this section.

Theorem 4.1 (KYP lemma for differential-algebraic systems)**.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$ be given. Further, let the Popov function $\Phi(s) \in \mathbb{K}(s)^{m \times m}$ be defined as in (4.1). Then the following statements hold true:*

(a) If there exists some $P \in \mathbb{K}^{n \times n}$ with (4.2), then

$$\Phi(\mathrm{i}\omega) \geq 0 \qquad \forall \omega \in \mathbb{R} \text{ with } \det(\mathrm{i}\omega E - A) \neq 0. \tag{4.3}$$

(b) If (4.3) and at least one of the two properties

(i) $[E, A, B]$ is behavioral sign-controllable and $\operatorname{rank}_{\mathbb{K}(s)} \Phi(s) = m$;

(ii) $[E, A, B]$ is behavioral controllable;

is satisfied, then there exists some $P \in \mathbb{K}^{n \times n}$ that solves the KYP inequality (4.2).

Statement (a) will be proven directly. For the proof of (b), we will first apply a feedback transformation leading to feedback equivalence form (2.3) and then we make use of the well-known result for ODE systems. The following lemma is the basis for this argumentation:

Lemma 4.2. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$ be given. Further, let the Popov function $\Phi(s) \in \mathbb{K}(s)^{m \times m}$ be defined as in (4.1). Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m \times n}$ be matrices leading to feedback equivalence form (2.3). Define*

$$\begin{aligned} &E_F = WET, \quad A_F = W(A + BF)T, \quad B_F = WB, \\ &Q_F = T^*(Q + SF + F^*S^* + F^*RF)T, \quad S_F = T^*(S + F^*R), \quad R_F = R, \end{aligned} \tag{4.4}$$

and partition

$$Q_F = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}, \quad S_F = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}. \tag{4.5}$$

according to the block structure of (2.3). Then the following statements hold true:

(a) The rational function

$$\Phi_F(s) = \begin{bmatrix} (-\overline{s}E_F - A_F)^{-1}B_F \\ I_m \end{bmatrix}^* \begin{bmatrix} Q_F & S_F \\ S_F^* & R_F \end{bmatrix} \begin{bmatrix} (sE_F - A_F)^{-1}B_F \\ I_m \end{bmatrix}$$

fulfills

$$\Phi_F(s) = \Theta_F^*(-\overline{s})\Phi(s)\Theta_F(s). \tag{4.6}$$

for $\Theta_F(s) = I_m + FT(sE_F - A_F)^{-1}B_F \in \mathbb{K}(s)^{m\times m}$*. Moreover, it holds*

$$\Phi_F(s) = \begin{bmatrix} (-\overline{s}I_{n_1} - A_{11})^{-1} B_1 \\ I_m \end{bmatrix}^* \begin{bmatrix} Q_{11} & S_1 - Q_{12}B_2 \\ S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \begin{bmatrix} (sI_{n_1} - A_{11})^{-1} B_1 \\ I_m \end{bmatrix}. \tag{4.7}$$

(b) For $P \in \mathbb{K}^{n\times n}$ *and*

$$P_F = W^{-*}PW^{-1} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} \in \mathbb{K}^{n\times n} \tag{4.8}$$

partitoned according to the block structure of the feedback equivalence form (2.3) it holds: $P \in \mathbb{K}^{n\times n}$ *solves the KYP inequality (4.2) if and only if* P *is Hermitian and*

$$\begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*P_{11} + S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \geq 0, \quad P_{11} = P_{11}^*. \tag{4.9}$$

Proof.

(a) The relation (4.6) is analogous to [14, Lem. 2.1] and has been proven in [43, Prop. 3.2.2]. The relation (4.7) follows from

$$\begin{bmatrix} (sE_F - A_F)^{-1}B_F \\ I_m \end{bmatrix} = \begin{bmatrix} (sI_{n_1} - A_{11})^{-1}B_1 \\ -B_2 \\ 0_{n_3\times m} \\ I_m \end{bmatrix}.$$

(b) Before we prove the equivalence, we first notice that

$$\begin{aligned} &\begin{bmatrix} A_F^*P_FE_F + E_F^*P_FA_F + Q_F & E_F^*P_FB_F + S_F \\ B_F^*P_FE_F + S_F^* & R_F \end{bmatrix} \\ &= \begin{bmatrix} T^* & T^*F^* \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}. \end{aligned}$$

Let $\mathcal{V}_{\mathrm{sys},F}$ be the system space of $[E_F, A_F, B_F]$. Using the above equation and Lemma 3.2, we obtain that for all Hermitian $P \in \mathbb{K}^{n\times n}$ it holds

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0 \iff \begin{bmatrix} A_F^*P_FE_F + E_F^*P_FA_F + Q_F & E_F^*P_FB_F + S_F \\ B_F^*P_FE_F + S_F^* & R_F \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys},F}} 0. \tag{4.10}$$

Further, we have

$$\begin{aligned} &\begin{bmatrix} A_F^*P_FE_F + E_F^*P_FA_F + Q_F & E_F^*P_FB_F + S_F \\ B_F^*P_FE_F + S_F^* & R_F \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^*P_{11} + P_{11}A_{11} + Q_{11} & P_{12} + Q_{12} & M_{13} & P_{11}B_1 + P_{12}B_2 + S_1 \\ P_{12}^* + Q_{12}^* & Q_{22} & M_{23} & S_2 \\ M_{13}^* & M_{23}^* & M_{33} & M_{34} \\ B_1^*P_{11} + B_2^*P_{12}^* + S_1^* & S_2^* & M_{34}^* & R \end{bmatrix}, \end{aligned} \tag{4.11}$$

for some $M_{13} \in \mathbb{K}^{n_1\times n_3}$, $M_{23} \in \mathbb{K}^{n_2\times n_3}$, $M_{33} \in \mathbb{K}^{n_3\times n_3}$, $M_{34} \in \mathbb{K}^{n_3\times m}$.

Now we show that if $P \in \mathbb{K}^{n\times n}$ solves the KYP inequality (4.2), then $P_{11} \in \mathbb{K}^{n_1\times n_1}$ solves (4.9). Therefore, let $\mathcal{V}_{\mathrm{sys},F}$ be the system space of $[E_F, A_F, B_F]$ and let $x_1 \in \mathbb{K}^{n_1}$, $u \in \mathbb{K}^m$. By (3.2) we see that

$$\begin{pmatrix} x_1 \\ -B_2 u \\ 0_{n_3\times 1} \\ u \end{pmatrix} \in \mathcal{V}_{\mathrm{sys},F}.$$

Then, by using (4.10) and (4.11), a simple calculation shows

$$\begin{aligned} 0 &\leq \begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A_F^* P_F E_F + E_F^* P_F A_F + Q_F & E_F^* P_F B_F + S_F \\ B_F^* P_F E_F + S_F^* & R_F \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} P_{11}A_{11} + A_{11}^* P_{11} + Q_{11} & P_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^* P_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix}, \end{aligned}$$

i.e., $P_{11} \in \mathbb{K}^{n_1\times n_1}$ fulfills (4.9).

Now we show the converse. Thus assume that $P_{11} \in \mathbb{K}^{n_1\times n_1}$ fulfills (4.9) and let $P_{12} \in \mathbb{K}^{n_1\times n_2}$, $P_{13} \in \mathbb{K}^{n_1\times n_3}$, $P_{22} \in \mathbb{K}^{n_2\times n_2}$, $P_{23} \in \mathbb{K}^{n_2\times n_3}$, and $P_{33} \in \mathbb{K}^{n_3\times n_3}$ be arbitrary given matrices. Furthermore, assume that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathcal{V}_{\mathrm{sys},F}$. Then, by (3.2), there exists some $x_1 \in \mathbb{K}^{n_1}$ with $x = \left(\begin{smallmatrix} x_1 \\ -B_2 u \\ 0_{n_3\times 1} \\ u \end{smallmatrix}\right) \in \mathcal{V}_{\mathrm{sys},F}$. By using (4.10) and (4.11), we now obtain

$$\begin{aligned} &\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{bmatrix} A_F^* P_F E_F + E_F^* P_F A_F + Q_F & E_F^* P_F B_F + S_F \\ B_F^* P_F E_F + S_F^* & R_F \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ u \end{pmatrix}^* \begin{bmatrix} P_{11}A_{11} + A_{11}^* P_{11} + Q_{11} & P_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^* P_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} \begin{pmatrix} x_1 \\ u \end{pmatrix} \geq 0. \end{aligned}$$

Then an application of (4.10) leads to the desired result.

□

Now we are able to prove Theorem 4.1.

Proof of Theorem 4.1.

(a) Assume that $P \in \mathbb{K}^{n\times n}$ fulfills (4.2). First note that

$$\begin{aligned} &\begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (-\overline{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \left(\begin{bmatrix} A^* \\ B^* \end{bmatrix} \begin{bmatrix} PE & 0 \end{bmatrix} + \begin{bmatrix} E^*P \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \right) \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \overset{(3.6)}{=} 0. \end{aligned} \tag{4.12}$$

Then, by using Lemma 3.5 and (4.12), we see that for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$ it holds

$$\begin{aligned} \Phi(\mathrm{i}\omega) &= \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix} \\ &\overset{(4.2)\,\&\,\text{Lem. } 3.5}{\geq} - \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1}B \\ I_m \end{bmatrix} \overset{(4.12)}{=} 0. \end{aligned} \tag{4.13}$$

(b) Now assume that (4.3) and one of the conditions (i) or (ii) holds true. By Proposition 2.9 there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ leading to feedback equivalence form (2.3). Then, with matrices as defined in (4.5) and (4.4), we obtain from Lemma 4.2 (a) that the function $\Phi_F(s) \in \mathbb{K}(s)^{m\times m}$ in (4.7) fulfills $\Phi_F(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ such that $\mathrm{i}\omega$ is not an eigenvalue of A_{11}. By further using Remark 2.10, we obtain that we are in the situation of the KYP lemma for ODE systems [14, Thm. 6.1 & Thm. 6.2]. Hence there exists some Hermitian $P_{11} \in \mathbb{K}^{n_1\times n_1}$ which fulfills the standard KYP inequality (4.9). From Lemma 4.2 (b) we obtain that

$$P = W^* \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W \in \mathbb{K}^{n\times n},$$

fulfills the KYP inequality (4.2).

$\square$

Remark 4.3.

(a) We obtain from Theorem 4.1 that, in the case where at least one of the assumptions (bi) or (bii) is fulfilled, the feasibility of the linear matrix inequality (4.2) is equivalent to the non-negativity property (4.3) of the Popov function.

(b) If the Popov function is positive semi-definite on the imaginary axis with $\operatorname{rank}_{\mathbb{K}(s)} \Phi(s) < m$ and the system is not behavioral sign-controllable, then the linear matrix inequality (4.2) might have an empty solution set. Counter-examples exist already in the ODE case, see [40, p. 88].

Next we present an alternative version of the KYP inequality, whose solvability will be proven to be equivalent to that of (4.2).

Proposition 4.4. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. If $Y \in \mathbb{K}^{n\times n}$ fulfills*

$$\begin{bmatrix} A^*Y + Y^*A + Q & Y^*B + S \\ B^*Y + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad E^*Y = Y^*E, \tag{4.14}$$

then there exists some $P \in \mathbb{K}^{n\times n}$ that solves the KYP inequality (4.2) with $E^*PE = E^*Y$.
On the other hand, if $P \in \mathbb{K}^{n\times n}$ solves the KYP inequality (4.2), then $Y = PE$ fulfills (4.14).

Proof. The second statement is trivial.

To prove the first assertion, assume that $Y \in \mathbb{K}^{n\times n}$ fulfills (4.14). Then we can apply a transformation to feedback equivalence form (2.3) and partition

$$Y_F = W^{-*}YT = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \in \mathbb{K}^{n\times n} \tag{4.15}$$

according to the block structure of the feedback equivalence form. Then, by $E_F^*Y_F = Y_F^*E_F$ and an argumentation analogous to the proof of Lemma 4.2 (b), we obtain that

$$\begin{bmatrix} A_{11}^*Y_{11} + Y_{11}A_{11} + Q_{11} & Y_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^*Y_{11} + S_1^* - B_2^*Q_{12}^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix} \geq 0, \quad Y_{11} = Y_{11}^*. \tag{4.16}$$

The equation $E_F^*Y_F = Y_F^*E_F$ leads to

$$E_F^*Y_F = \begin{bmatrix} Y_{11} & 0 & Y_{21}E_{23} + Y_{31}E_{33} \\ 0 & 0 & 0 \\ E_{23}^*Y_{21} + E_{33}^*Y_{31} & 0 & E_{23}^*Y_{23} + E_{33}^*Y_{33} \end{bmatrix},$$

where $E_{23}^*Y_{23} + E_{33}^*Y_{33}$ is Hermitian. The latter implies that there exist $P_{22} = P_{22}^* \in \mathbb{K}^{n_2\times n_2}$, $P_{23} \in \mathbb{K}^{n_2\times n_3}$ and $P_{33} = P_{33}^* \in \mathbb{K}^{n_3\times n_3}$ such that

$$E_{23}^*Y_{23} + E_{33}^*Y_{33} = \begin{bmatrix} E_{23}^* & E_{33}^* \end{bmatrix} \begin{bmatrix} P_{22} & P_{23} \\ P_{23}^* & P_{33} \end{bmatrix} \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix}.$$

Then, a simple calculation shows that

$$P = W^* \begin{bmatrix} Y_{11} & Y_{21}^* & Y_{31}^* \\ Y_{21} & P_{22} & P_{23} \\ Y_{31} & P_{32} & P_{33} \end{bmatrix} W$$

fulfills $E^*PE = E^*Y$. Further, (4.16) together with Lemma 4.2 (b) implies that P fulfills the KYP inequality (4.2). $\square$

5 Lur'e Equations, stabilizing and extremal solutions

In this section our focus is on particular solutions of the KYP inequality. More precisely, for $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$, we consider the *Lur'e equation*

$$\begin{bmatrix} A^*XE + E^*XA + Q & E^*XB + S \\ B^*XE + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*, \tag{5.1a}$$

which has to be solved for a triple $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ such that

$$\operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} = n + q. \tag{5.1b}$$

Note that the number $q \in \mathbb{N}_0$ is not given beforehand; it is, loosely speaking, part of the solution. We will show that $q = \operatorname{rank}_{\mathbb{K}(s)} \Phi(s)$. It follows immediately that $X \in \mathbb{K}^{n\times n}$ in (5.1) is a solution of the KYP inequality (4.2). We further consider the following particular solutions.

Definition 5.1 (Stabilizing and anti-stabilizing solution of the Lur'e equation)**.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. Then a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ of the Lur'e equation (5.1) is called

(i) *stabilizing*, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}_+; \tag{5.2}$$

(ii) *anti-stabilizing*, if

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = n + q \quad \forall \lambda \in \mathbb{C}_-. \tag{5.3}$$

Remark 5.2 (Solutions of Lur'e equations). Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$ and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given.

(a) In the ODE case, the system space reads $\mathcal{V}_{\text{sys}} = \mathbb{K}^{n+m}$. In this case, the Lur'e equation (5.1) reduces to

$$\begin{aligned} A^*X + XA + Q &= K^*K, \quad X = X^*, \\ XB + S &= K^*L, \\ R &= L^*L \end{aligned} \tag{5.4}$$

together with (5.1b) for $E = I_n$. This type has been treated in [2, 38]. Our notions of stabilizing and anti-stabilizing solutions also coincide with the corresponding notions in [2, 38].

(b) We briefly present the transformation of solutions of the Lur'e equation under feedback action: To this end, consider $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{n\times m}$ such that $sE - (A + BF)$ is regular, and let E_F, A_F, B_F, Q_F, S_F, and R_F be defined as in (4.4). Then (X, K, L) is a (stabilizing, anti-stabilizing) solution of (5.1) if and only if

$$(X_F, K_F, L_F) = (W^{-*}XW^{-1}, KT + LFT, L) \tag{5.5}$$

solves the Lur'e equation associated to E_F, A_F, B_F, Q_F, S_F, and R_F. The validity of

$$\begin{bmatrix} A_F^*X_FE_F + E_F^*X_FA_F + Q_F & E_F^*X_FB_F + S_F \\ B_F^*X_FE_F + S_F^* & R_F \end{bmatrix} =_{\mathcal{V}_{\text{sys},F}} \begin{bmatrix} K_F^* \\ L_F^* \end{bmatrix} \begin{bmatrix} K_F & L_F \end{bmatrix}, \quad X_F = X_F^* \tag{5.6}$$

follows by the same argumentation as in Step 2 in the proof of Theorem 4.1 (b). The additional condition (5.1b) ((5.2), (5.3)) simply follows by

$$\begin{bmatrix} -sE_F + A_F & B_F \\ K_F & L_F \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix}. \tag{5.7}$$

In the following we analyze the existence of (stabilizing, anti-stabilizing) solutions of Lur'e equations for differential-algebraic equations.

Theorem 5.3 (Existence of solutions of Lur'e equations)**.** *Let a system $[E,A,B]\in\Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\rm sys}\subset\mathbb{K}^{n+m}$ be given, and let $Q=Q^*\in\mathbb{K}^{n\times n}$, $S\in\mathbb{K}^{n\times m}$, $R=R^*\in\mathbb{K}^{m\times m}$. Assume that the KYP inequality (4.2) has a solution $P\in\mathbb{K}^{n\times n}$.*

(a) If $[E,A,B]$ has no uncontrollable modes on $\mathrm{i}\mathbb{R}$, then the Lur'e equation (5.1) has a solution.

(b) If $[E,A,B]$ is behavioral stabilizable, then the Lur'e equation (5.1) has a stabilizing solution.

(c) If $[E,A,B]$ is behavioral anti-stabilizable, then the Lur'e equation (5.1) has an anti-stabilizing solution.

The proof is based on the following auxiliary result, which can be seen as a version of Lemma 4.2 (b) for Lur'e equations:

Lemma 5.4. *Let $[E,A,B]\in\Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\rm sys}\subset\mathbb{K}^{n+m}$, and $Q=Q^*\in\mathbb{K}^{n\times n}$, $S\in\mathbb{K}^{n\times m}$, $R=R^*\in\mathbb{K}^{m\times m}$ be given. Let $W,T\in\mathrm{Gl}_n(\mathbb{K})$ and $F\in\mathbb{K}^{m\times n}$ be matrices leading to feedback equivalence form (2.3). Define the matrices E_F, A_F, B_F, Q_F, S_F, and R_F as in (4.4) and partition these matrices as in (4.5). Then for $(X,K,L)\in\mathbb{K}^{n\times n}\times\mathbb{K}^{q\times n}\times\mathbb{K}^{q\times m}$ and*

$$X_F=W^{-*}XW^{-1}=\begin{bmatrix}X_{11}&X_{12}&X_{13}\\X_{12}^*&X_{22}&X_{23}\\X_{13}^*&X_{23}^*&X_{33}\end{bmatrix}\in\mathbb{K}^{n\times n},\quad K_F=(K+LF)T=\begin{bmatrix}K_1&K_2&K_3\end{bmatrix}\in\mathbb{K}^{q\times n},\tag{5.8}$$

partitioned according to the block structure of the feedback equivalence form (2.3) it holds: $(X,K,L)\in\mathbb{K}^{n\times n}\times\mathbb{K}^{q\times n}\times\mathbb{K}^{q\times m}$ is a (stabilizing, anti-stabilizing) solution of the Lur'e equation (5.1) if and only if X_{11} is Hermitian with

$$\begin{bmatrix}A_{11}^*X_{11}+X_{11}A_{11}+Q_{11}&X_{11}B_1+S_1-Q_{12}B_2\\B_1^*X_{11}+S_1^*-B_2^*Q_{12}^*&B_2^*Q_{22}B_2-B_2^*S_2-S_2^*B_2+R\end{bmatrix}=\begin{bmatrix}K_1^*\\(L-K_2B_2)^*\end{bmatrix}\begin{bmatrix}K_1&L-K_2B_2\end{bmatrix}\tag{5.9a}$$

and

$$\begin{gathered}\mathrm{rank}_{\mathbb{K}(s)}\begin{bmatrix}-sI_{n_1}+A_{11}&B_1\\K_1&L-K_2B_2\end{bmatrix}=n+q,\\\left(\mathrm{rank}\begin{bmatrix}-\lambda I_{n_1}+A_{11}&B_1\\K_1&L-K_2B_2\end{bmatrix}=n+q\ \forall\,\lambda\in\mathbb{C}_+,\quad\mathrm{rank}\begin{bmatrix}-\lambda I_{n_1}+A_{11}&B_1\\K_1&L-K_2B_2\end{bmatrix}=n+q\ \forall\,\lambda\in\mathbb{C}_-\right).\end{gathered}\tag{5.9b}$$

Proof. The statement follows by an argumentation which is analogous to that in the proof of Lemma 4.2 (b), and the fact that for all $\lambda\in\mathbb{C}$ it holds (by invoking $\mathrm{rank}(\lambda E_{33}-I_{n_3})=n_3$ for all $\lambda\in\mathbb{C}$)

$$\begin{aligned}\mathrm{rank}\begin{bmatrix}-\lambda E+A&B\\K&L\end{bmatrix}&=\mathrm{rank}\begin{bmatrix}W&0\\0&I_q\end{bmatrix}\begin{bmatrix}-\lambda E+A&B\\K&L\end{bmatrix}\begin{bmatrix}T&0\\FT&I_m\end{bmatrix}\\&=\mathrm{rank}\begin{bmatrix}-\lambda I_{n_1}+A_{11}&0&0&B_1\\0&I_{n_2}&-\lambda E_{23}&B_2\\0&0&-\lambda E_{33}+I_{n_3}&0\\K_1&K_2&K_3&L\end{bmatrix}\begin{bmatrix}I_{n_1}&0&0&0\\0&-B_2&I_{n_2}&0\\0&0&0&I_{n_3}\\0&I_m&0&0\end{bmatrix}\\&=\mathrm{rank}\begin{bmatrix}-\lambda I_{n_1}+A_{11}&B_1&0&0\\0&0&I_{n_2}&-\lambda E_{23}\\0&0&0&-\lambda E_{33}+I_{n_3}\\K_1&L-K_2B_2&K_2&K_3\end{bmatrix}\\&=n_2+n_3+\mathrm{rank}\begin{bmatrix}-\lambda I_{n_1}+A_{11}&B_1\\K_1&L-K_2B_2\end{bmatrix}.\end{aligned}\tag{5.10}$$

□

Proof of Theorem 5.3. Assume that $W, T \in \mathrm{Gl}_n(\mathbb{K})$, $F \in \mathbb{K}^{m\times n}$ lead to feedback equivalence form (2.3), define the matrices Q_F, S_F, and R_F as in (4.4), and partition Q_F and S_F as in (4.5). Suppose that $P \in \mathbb{K}^{n\times n}$ fulfills (4.2). Consider $P_F = W^{-*}PW^{-1}$ and partitioned according to the block structure of the feedback equivalence form (2.3). Then, by Lemma 4.2 (b), $P_{11} \in \mathbb{K}^{n_1\times n_1}$ fulfills the standard KYP inequality (4.9). Then the following holds true:

(a) If $[E, A, B]$ has no uncontrollable modes on $\mathrm{i}\mathbb{R}$, then $[I_{n_1}, A_{11}, B_1]$ has no uncontrollable modes on $\mathrm{i}\mathbb{R}$ by Proposition 2.9 (c). Then [38, Lem. 12] implies the existence of a triple $(X_{11}, K_1, L_1) \in \mathbb{K}^{n_1\times n_1} \times \mathbb{K}^{q\times n_1} \times \mathbb{K}^{q\times m}$ such that

$$\begin{bmatrix} A_{11}^* X_{11} + X_{11}A_{11} + Q_{11} & X_{11}B_1 + S_1 - Q_{12}B_2 \\ B_1^* X_{11} + S_1^* - B_2^* Q_{12}^* & B_2^* Q_{22} B_2 - B_2^* S_2 - S_2^* B_2 + R \end{bmatrix} = \begin{bmatrix} K_1^* \\ L_1^* \end{bmatrix} \begin{bmatrix} K_1 & L_1 \end{bmatrix}, \quad X_{11} = X_{11}^*. \tag{5.11}$$

Now define

$$X = W^* \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W, \quad K = \begin{bmatrix} K_1 & 0 & 0 \end{bmatrix} T^{-1} - LF, \quad L = L_1 \tag{5.12}$$

Lemma 5.4 then implies that (X, K, L) solves the Lur'e equation (5.1).

(b) If $[E, A, B]$ is behavioral stabilizable, then $[I_{n_1}, A_{11}, B_1]$ is stabilizable by Remark 2.10. Then, by [38, Thm. 15], the standard Lur'e equation (5.11) has a stabilizing solution $(X_{11}, K_1, L_1) \in \mathbb{K}^{n_1\times n_1} \times \mathbb{K}^{q\times n_1} \times \mathbb{K}^{q\times m}$. Define the triple (X, K, L) as in (5.12) and let $\lambda \in \mathbb{C}_+$. Then Lemma 5.4 implies that (X, K, L) is a stabilizing solution of the Lur'e equation (5.1).

(c) The proof of statement (c) is analogous to (b) (by using [38, Thm. 16] instead of [38, Thm. 15]).

□

In the following theorem we will show that the stabilizing and anti-stabilizing solution of the Lur'e equation are extremal solutions in terms of definiteness of E^*XE.

Theorem 5.5. *Let a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$, the space of consistent differential variables $\mathcal{V}_{\mathrm{diff}} \subset \mathbb{K}^n$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. Assume that the KYP inequality (4.2) has a solution $P \in \mathbb{K}^{n\times n}$.*

(a) If (X, K, L) is a stabilizing solution of the Lur'e equation (5.1), then

$$E^*XE \geq_{\mathcal{V}_{\mathrm{diff}}} E^*PE.$$

(b) If (X, K, L) is an anti-stabilizing solution of the Lur'e equation (5.1), then

$$E^*PE \geq_{\mathcal{V}_{\mathrm{diff}}} E^*XE.$$

Proof. Assume that $P \in \mathbb{K}^{n\times n}$ fulfills (4.2). Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$, and $F \in \mathbb{K}^{m\times n}$ be matrices leading to feedback equivalence form (2.3). Consider $P_F = W^{-*}PW^{-1}$ and partition it as in (4.8). Then, by Lemma 4.2 (b), the standard KYP lemma (4.9) holds true.

(a) Assume that (X, K, L) is a stabilizing solution of the Lur'e equation (5.1). Define (X_F, K_F, L_F) as in (5.5), and partition as in (5.8) according to the block structure of the feedback equivalence form (2.3). Then, by Lemma 4.2(b), $(X_{11}, K_1, L - B_2K_2)$ fulfills the standard Lur'e equation (5.9). Then we can apply the corresponding result for ODE systems [38, Thm. 15] to see that $X_{11} \geq P_{11}$. Now assume that $x \in \mathcal{V}_{\mathrm{diff}}$. By Proposition 2.9 (b), we have

$$T^{-1}x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

for $x_1 \in \mathbb{K}^{n_1}$, $x_2 \in \mathbb{K}^{n_2}$ and $x_3 \in \ker \left[\begin{smallmatrix} E_{23} \\ E_{33} \end{smallmatrix}\right]$, and thus

$$\begin{aligned} x^*E^*XEx =& (T^{-1}x)^* E_F^* X_F E_F (T^{-1}x) \\ =& \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^* \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_{23}^* & E_{33}^* \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^* X_{11} x_1, \end{aligned}$$

and analogously,

$$x^* E^* P E x = x_1^* P_{11} x_1.$$

Then the inequality $X_{11} \geq P_{11}$ gives rise to $E^* X E \geq_{\mathcal{V}_{\text{diff}}} E^* P E$.

(b) The proof of statement (b) is analogous to (a) (by using [38, Thm. 16] instead of [38, Thm. 15]).

□

Corollary 5.6. *Under the assumption and notation of Theorem 5.5, the following holds true: If (X_1, K_1, L_1) and (X_2, K_2, L_2) are stabilizing (anti-stabilizing) solutions of (5.1), then*

$$E^* X_1 E =_{\mathcal{V}_{\text{diff}}} E^* X_2 E.$$

Proof. Assume that (X_1, K_1, L_1) and (X_2, K_2, L_2) are stabilizing solutions of (5.1). Then X_1, X_2 are solutions of the KYP inequality (4.2) and hence, by Theorem 5.5 (a), we have

$$E^* X_1 E \geq_{\mathcal{V}_{\text{diff}}} E^* X_2 E \geq_{\mathcal{V}_{\text{diff}}} E^* X_1 E.$$

This implies the result for stabilizing solutions. The case of anti-stabilizing solutions can be readily shown by turning the inequality symbols in the above argumentation. □

Remark 5.7. Let a system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$, the space of consistent differential variables $\mathcal{V}_{\text{diff}} \subset \mathbb{K}^n$ and $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$ be given. Let the Popov function $\Phi(s) \in \mathbb{K}(s)^{m \times m}$ be defined as in (4.1).

(a) If $[E, A, B]$ is impulse controllable, then the inequalities in Theorem 5.5 clearly reduce to $E^* X E \geq E^* P E$ and $E^* P E \geq E^* X E$, respectively.

(b) Now we show that solutions of Lur'e equations are rank-minimizing in a certain sense and specify the number $q \in \mathbb{N}_0$ is a solution $(X, K, L) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{q \times n} \times \mathbb{K}^{q \times m}$ of the Lur'e equation (5.1): If P solves the KYP inequality (4.2), then we can find matrices $M \in \mathbb{K}^{l \times n}$, $N \in \mathbb{K}^{l \times m}$ such that

$$\begin{bmatrix} A^* P E + E^* P A + Q & E^* P B + S \\ B^* P E + S^* & R \end{bmatrix} =_{\mathcal{V}_{\text{sys}}} \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix}. \tag{5.13}$$

Assume that $\omega \in \mathbb{R}$ such that $\det(\mathrm{i}\omega E - A) \neq 0$. Then the Popov function fulfills

$$\begin{aligned}
\Phi(\mathrm{i}\omega) &= \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1} B \\ I_m \end{bmatrix} \\
&\overset{\text{Lem. 3.5\&(5.1a)}}{=} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix} \\
&\quad - \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^* X E + E^* X A & E^* X B \\ B^* X E & 0 \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix} \\
&\overset{(4.12)}{=} \left(\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix} \right)^* \left(\begin{bmatrix} M & N \end{bmatrix} \begin{bmatrix} (\mathrm{i}\omega E - A)^{-1} B \\ I_m \end{bmatrix} \right) = Z^*(\mathrm{i}\omega) Z(\mathrm{i}\omega),
\end{aligned}$$

where $Z(\mathrm{i}\omega) = N + M(\mathrm{i}\omega E - A)^{-1} B$. Thereby we obtain $l \geq \operatorname{rank}_{\mathbb{K}(s)} \Phi(s)$. On the other hand, for a solution of the Lur'e equation (5.1) an analogous computation shows that $W(s) = L + K(sE - A)^{-1} B$ fulfills

$$\Phi(\mathrm{i}\omega) = W^*(\mathrm{i}\omega) W(\mathrm{i}\omega) \tag{5.14}$$

for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. By a simple row transformation in (5.1b), we obtain $\operatorname{rank}_{\mathbb{K}(s)} W(s) = q$, which altogether gives

$$q = \operatorname{rank}_{\mathbb{K}(s)} W(s) = \operatorname{rank}_{\mathbb{K}(s)} \Phi(s) \leq l.$$

The consequence to twofold: First, the number q which specifies the number of rows of K and L in a solution of the Lur'e equation equals to $\operatorname{rank}_{\mathbb{K}(s)} \Phi(s)$. Second, the Lur'e equation can be regarded as a KYP inequality in which the rank of the right hand side of (5.13) is minimized.

(c) A representation (5.14) in which $W(s)$ is an *outer transfer function* [20] is called *spectral factorization* (see e.g. [48, Sec. 13.4] for the ODE case).

We finally present two alternative ways for the reformulation of Lur'e equations (5.1). The first alternative way is a version of Proposition 4.4 for Lur'e equations. The proof is completely analogous to that of Proposition 4.4, and thus it is omitted.

Proposition 5.8. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\rm sys} \subset \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. If $(H, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ fulfills*

$$\begin{bmatrix} A^*H + H^*A + Q & H^*B + S \\ B^*H + S^* & R \end{bmatrix} =_{\mathcal{V}_{\rm sys}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad E^*H = H^*E. \tag{5.15}$$

then there exists some $X \in \mathbb{K}^{n\times n}$ that fulfills (5.1a) with $E^*XE = E^*H$.
On the other hand, if $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ solves (5.1), then (H, K, L) with $H = XE$ fulfills (5.15).

We further show that the system itself can be remodeled such that an impulse controllable system with the same behavior is obtained. The corresponding solution set of the KYP inequality and the Lur'e equation will further be the same as for the original system.

Theorem 5.9. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\rm sys} \subset \mathbb{K}^{n+m}$, the space of consistent differential variables $\mathcal{V}_{\rm diff} \subset \mathbb{K}^n$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. Assume that $W, T \in \mathrm{Gl}_n(\mathbb{K})$, and $F \in \mathbb{K}^{m\times n}$ are transformation matrices leading to feedback equivalence form (2.3). Define the projector*

$$\Pi = W^{-1} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W \in \mathbb{K}^{n\times n}.$$

Then we have

$$\operatorname{im} \Pi = E\mathcal{V}_{\rm diff}, \tag{5.16}$$

and the following statements hold true:

(a) $[\Pi E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is impulse controllable and

$$\mathfrak{B}_{[E,A,B]} = \mathfrak{B}_{[\Pi E,A,B]}.$$

In particular, the system space of $[\Pi E, A, B]$ is $\mathcal{V}_{\rm sys}$.

(b) $P \in \mathbb{K}^{n\times n}$ fulfills the KYP inequality (4.2) if and only if

$$\begin{bmatrix} A^*P\Pi E + E^*\Pi^*PA + Q & E^*\Pi^*PB + S \\ B^*P\Pi E + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\rm sys}} 0, \quad P = P^*. \tag{5.17}$$

(c) $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ fulfills the Lur'e equation (5.1a) if and only if

$$\begin{bmatrix} A^*X\Pi E + E^*\Pi^*XA + Q & E^*\Pi^*XB + S \\ B^*X\Pi E + S^* & R \end{bmatrix} =_{\mathcal{V}_{\rm sys}} \begin{bmatrix} K^* \\ L^* \end{bmatrix} \begin{bmatrix} K & L \end{bmatrix}, \quad X = X^*. \tag{5.18}$$

Furthermore, it holds

$$\operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ K & L \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -\lambda \Pi E + A & B \\ K & L \end{bmatrix} \quad \forall \lambda \in \mathbb{C}. \tag{5.19}$$

Proof. By Proposition 2.9 (b), we have

$$E\mathcal{V}_{\rm diff} = ET\left(\mathbb{K}^{n_1+n_2} \times \ker \begin{bmatrix} E_{23} \\ E_{33} \end{bmatrix}\right) = W^{-1}\left(\mathbb{K}^{n_1} \times \{0_{(n_2+n_3)\times 1}\}\right) \subset \mathbb{K}^n,$$

and thus (5.16) is immediate. Now define

$$\Pi_F := W\Pi W^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and E_F, A_F, B_F, Q_F, S_F, and R_F as in (4.4).

(a) We have

$$s\Pi_F E_F - A_F = \begin{bmatrix} sI_{n_1} - A_{11} & 0 & 0 \\ 0 & -I_{n_2} & 0 \\ 0 & 0 & -I_{n_3} \end{bmatrix}, \quad B_F = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}.$$

Then Proposition 2.3 (a) and Remark 2.8 imply that $[\Pi E, A, B]$ is impulse controllable. A straightforward calculation further shows that

$$\begin{aligned} \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, u\right) \in \mathfrak{B}_{[\Pi_F E_F, A_F, B_F]} \quad &\iff \quad x_2 = -B_2 u, \quad x_3 = 0, \quad (x_1, u) \in \mathfrak{B}_{[I_n, A_{11}, B_1]} \\ &\iff \quad \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, u\right) \in \mathfrak{B}_{[E_F, A_F, B_F]}. \end{aligned}$$

Then the statement follows from

$$\begin{aligned} (x, u) \in \mathfrak{B}_{[E,A,B]} \quad &\iff (T^{-1}x, u - FT^{-1}x) \in \mathfrak{B}_{[E_F, A_F, B_F]} \\ &\iff (T^{-1}x, u - FT^{-1}x) \in \mathfrak{B}_{[\Pi_F E_F, A_F, B_F]} \quad \iff \quad (x, u) \in \mathfrak{B}_{[\Pi E, A, B]}. \end{aligned}$$

(b) Let $P \in \mathbb{K}^{n \times n}$ and partition $P_F = W^{-*} P W^{-1}$ as in (4.8) and Q_F, S_F as in (4.5). Then it follows analogous to the proof of Lemma 4.2 (b) that P fulfills the KYP inequality (5.17) if, and only if, $P_{11} \in \mathbb{K}^{n_1 \times n_1}$ fulfills (4.9). The latter is, by Lemma 4.2 (b), equivalent to P fulfilling the KYP inequality (4.2).

(c) The proof of equivalence between (5.1a) and (5.18) is analogous to that of (c). Equation (5.19) follows by a computation analogous to (5.10).

□

Remark 5.10. Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with space of consistent differential variables $\mathcal{V}_{\text{diff}} \subset \mathbb{K}^n$ be given. Let $\Pi \in \mathbb{K}^{n \times n}$ be a projector with (5.16). Theorem 5.9 (a) states that the multiplication of E from the left with Π does not change the behavior. Such a procedure is called *index reduction*, and has, e.g., been done in [25] for linear systems with varying matrix coefficient. In fact, in the constant coefficient case, index reduction by replacing $[E, A, B]$ by $[\Pi E, A, B]$ is equivalent to the reduction of the strangeness index presented in [25, Sec. 2].

6 Even Matrix Pencils

Here we show that solutions of Lur'e equations correspond to certain deflating subspaces of the matrix pencil

$$s\mathcal{E} - \mathcal{A} = \begin{bmatrix} 0 & -s\Pi E + A & B \\ sE^*\Pi^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \in \mathbb{K}[s]^{2n+m \times 2n+m}, \tag{6.1}$$

where $\Pi \in \mathbb{K}^{n \times n}$ is defined as in (5.9). This generalizes the well-known fact that solutions of algebraic Riccati equations (1.5) correspond to certain invariant subspaces of the associated *Hamiltonian matrix* [30, Chap. 7]. This has been considered in [38] for Lur'e equations for ODE systems. This consideration is the basis for their numerical solution [34, 35].

The following concept generalizes the notion of invariant subspaces to matrix pencils. Moreover, we define the term of $\mathcal{E}$-neutrality of a subspace [38], which generalizes the notion of an isotropic subspace (typically in the context of Hamiltonian matrices).

Definition 6.1 (Basis matrix, deflating subspace, $\mathcal{E}$-neutrality)**.**

(a) A matrix $Y \in \mathbb{K}^{n \times r}$ is called a *basis matrix* for a subspace $\mathcal{Y} \subset \mathbb{K}^n$ if it has full column rank and $\operatorname{im} Y = \mathcal{Y}$.

(b) A subspace $\mathcal{Y} \subset \mathbb{C}^n$ is called *(right) deflating subspace deflating subspace* for the pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{K}[s]^{m\times n}$ if, for a basis matrix $Y \in \mathbb{K}^{n\times k}$ of $\mathcal{Y}$, there exists some $l \in \mathbb{N}_0$, a matrix $Z \in \mathbb{K}^{m\times l}$ and a pencil $s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} \in \mathbb{K}[s]^{l\times k}$ with $\operatorname{rank}_{\mathbb{K}(s)} \big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big) = l$, such that

$$(s\mathcal{E} - \mathcal{A})Y = Z\big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big).$$

(c) A subspace $\mathcal{Y} \subset \mathbb{K}^n$ is called $\mathcal{E}$*-neutral* if $y_1^*\mathcal{E}y_2 = 0$ for all y_1, $y_2 \in \mathcal{Y}$.

The matrix pencil (6.1) has the special property that it is *even*, i.e., $s\mathcal{E} - \mathcal{A} = -s\mathcal{E}^* - \mathcal{A}^*$. This structure has been analyzed in [12,38] in the context of Lur'e equations for ODE systems. The following result shows that solutions of the Lur'e equation (5.1) define $\mathcal{E}$-neutral deflating subspaces of the even matrix pencil (6.1).

Theorem 6.2. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\text{sys}} \subset \mathbb{K}^{n+m}$ and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. Furthermore, let $\Pi \in \mathbb{K}^{n\times n}$ be the projector as defined in (5.16). Then the following two statements are equivalent:*

(a) The Lur'e equation (5.1) has a solution $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$.

(b) It holds $\Phi(\mathrm{i}\omega) \geq 0$ for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A)$ and there exist Y_μ, $Y_x \in \mathbb{K}^{n\times n+m}$, $Y_u \in \mathbb{K}^{m\times n+m}$, Z_μ, $Z_x \in \mathbb{K}^{n\times n+q}$, $Z_u \in \mathbb{K}^{m\times n+q}$ such that for

$$Y = \begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}, \quad Z = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u \end{bmatrix}, \tag{6.2}$$

the following holds true:

(i) the space $\operatorname{im} Y$ is $n+m$-dimensional and $\mathcal{E}$-neutral;

(ii) $\mathcal{V}_{\text{sys}} \subset \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

(iii) $\operatorname{rank} \Pi E Y_x = n_1$;

(iv) there exist $\widetilde{\mathcal{E}}, \widetilde{\mathcal{A}} \in \mathbb{K}^{n+q\times n+m}$ with $\operatorname{rank}_{\mathbb{K}(s)} \big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big) = n+q$, such that

$$(s\mathcal{E} - \mathcal{A})Y = Z\big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big). \tag{6.3}$$

Proof. First we prove that (a) implies (b): By Proposition 2.9 (d), the feedback equivalence form of $[\Pi E, A, B]$ has the form

$$E_F = W\Pi E T = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_F = W(A + BF)T = \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B_F = WB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{6.4}$$

where W, $T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$. Accordingly partition the mtrices

$$\begin{aligned} Q_F &= T^*(Q + SF + F^*S^* + F^*RF)T = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}, \quad S_F = T^*(S + F^*R) = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \quad R_F = R, \\ X_F &= W^{-*}XW = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \quad K_F = (K + LF)T = \begin{bmatrix} K_1 & K_2 \end{bmatrix}. \end{aligned} \tag{6.5}$$

Assume that $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ solves the Lur'e equation (5.1). It follows from Theorem 5.9 (c) and Lemma 5.4 that $(X_{11}, K_1, L - K_2B_2)$ is a solution of the Lur'e equation (5.9a). From [38, Thm. 11] we deduce

$$\underbrace{\begin{bmatrix} 0 & -sI_{n_1}+A_{11} & B_1 & 0 & 0 \\ sI_{n_1}+A_{11}^* & Q_{11} & -Q_{12}B_2+S_1 & 0 & 0 \\ B_1^* & -B_2^*Q_{12}^*+S_1^* & B_2^*Q_{22}B_2-B_2^*S_2-S_2^*B_2+R & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{n_2} & 0 \end{bmatrix}}_{=:s\widehat{\mathcal{E}}_F-\widehat{\mathcal{A}}_F} \underbrace{\begin{bmatrix} X_{11} & 0 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_{n_2} \\ 0 & 0 & 0 \end{bmatrix}}_{=:\widehat{Y}_F}$$

$$= \underbrace{\begin{bmatrix} I_{n_1} & 0 & 0 \\ -X_{11} & K_1^* & 0 \\ 0 & (L-K_2B_2)^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}}_{=:\widehat{Z}_F} \underbrace{\begin{bmatrix} -sI_{n_1}+A_{11} & B_1 & 0 \\ K_1 & L-K_2B_2 & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix}}_{=:s\widetilde{\mathcal{E}}_F-\widetilde{\mathcal{A}}_F}, \tag{6.6}$$

where $\operatorname{im}\widehat{Y}_F$ is an $n+m$-dimensional $\widehat{\mathcal{E}}_F$-neutral deflating subspace. Define the matrices

$$\widehat{U} := \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & -Q_{12}^* & Q_{22}B_2-S_2 & -\frac{1}{2}Q_{22} & I_{n_2} \\ 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & -B_2 & I_{n_2} & 0 \\ 0 & 0 & I_m & 0 & 0 \end{bmatrix}, \quad \widehat{V} := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n_2} & B_2 \end{bmatrix}, \quad \widehat{W} := \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_{n_2} & K_2 \end{bmatrix}. \tag{6.7}$$

Then we obtain $\big(s\mathcal{E}_F-\mathcal{A}_F\big)Y_F = Z_F\big(s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}\big)$ for

$$\begin{aligned} s\mathcal{E}_F-\mathcal{A}_F &:= \widehat{U}^{-*}\big(s\widehat{\mathcal{E}}_F-\widehat{\mathcal{A}}_F\big)\widehat{U}^{-1} = \begin{bmatrix} 0 & 0 & -sI_{n_1}+A_{11} & 0 & B_1 \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ sI_{n_1}+A_{11}^* & 0 & Q_{11} & Q_{12} & S_1 \\ 0 & I_{n_2} & Q_{12}^* & Q_{22} & S_2 \\ B_1^* & B_2^* & S_1^* & S_2^* & R \end{bmatrix}, \\ Y_F &:= \widehat{U}\widehat{Y}_F\widehat{V} = \begin{bmatrix} X_{11} & 0 & 0 \\ -Q_{12}^* & -\frac{1}{2}Q_{22} & \frac{1}{2}Q_{22}B_2-S_2 \\ I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_m \end{bmatrix} =: \begin{bmatrix} Y_{\mu,1} \\ Y_{\mu,2} \\ Y_{x,1} \\ Y_{x,2} \\ Y_{u,F} \end{bmatrix}, \qquad (6.8) \\ Z_F &:= \widehat{U}^{-*}\widehat{Z}_F\widehat{W}^{-1}, \\ s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}} &:= \widehat{W}\big(s\widetilde{\mathcal{E}}_F-\widetilde{\mathcal{A}}_F\big)\widehat{V} = \begin{bmatrix} -sI_{n_1}+A_{11} & 0 & B_1 \\ 0 & I_{n_2} & B_2 \\ K_1 & K_2 & L \end{bmatrix}. \qquad (6.9) \end{aligned}$$

Note that $\operatorname{im}Y_F$ is and $n+m$-dimensional $\mathcal{E}_F$-neutral subspace. With $Y_{x,F} := \left[\begin{smallmatrix} Y_{x,1} \\ Y_{x,2} \end{smallmatrix}\right]$ and $\mathcal{V}_{\mathrm{sys},F}$ as in (3.1), we see that $\mathcal{V}_{\mathrm{sys},F} \subset \operatorname{im}\left[\begin{smallmatrix} Y_{x,1} \\ Y_{x,2} \\ Y_{u,1} \end{smallmatrix}\right]$ and $\operatorname{rank}Y_{x,1} = n_1$. In other words, all the properties in (b) hold true for the Lur'e equation corresponding to the system in feedback equivalence form. Since with

$$U := \begin{bmatrix} W^* & 0 & 0 \\ 0 & T & 0 \\ 0 & FT & I_m \end{bmatrix} \tag{6.10}$$

we have $(s\mathcal{E}-\mathcal{A})Y = Z\big(s\widetilde{\mathcal{E}}-\widetilde{\mathcal{A}}\big)$ for $s\mathcal{E}-\mathcal{A} = U^{-*}\big(s\mathcal{E}_F-\mathcal{A}_F\big)U^{-1}$, $Y = UY_F$ and $Z = U^{-*}Z_F$. In other words, property (b) (iv) holds true. We further have:

(i) $\operatorname{im}Y$ is an $n+m$-dimensional $\mathcal{E}$-neutral deflating subspace;

(ii) $\mathcal{V}_{\rm sys} = \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \cdot \mathcal{V}_{{\rm sys},F} \subset \operatorname{im} \begin{bmatrix} T & 0 \\ FT & I_m \end{bmatrix} \cdot \begin{bmatrix} Y_{x,F} \\ Y_{u,F} \end{bmatrix} = \operatorname{im} \begin{bmatrix} Y_x \\ Y_u \end{bmatrix}$;

(iii) $\operatorname{rank} \Pi E Y_x = \operatorname{rank} W^{-1} E_F T^{-1} T Y_{x,F} = \operatorname{rank} \begin{bmatrix} Y_{x,1} \\ 0 \end{bmatrix} = n_1$.

Altogether, we obtain that (b) holds.

Now we prove that (b) implies (a): Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and $F \in \mathbb{K}^{m\times n}$ be the transformation matrices leading to feedback equivalence form of $[\Pi E, A, B] \in \Sigma_{n,m}(\mathbb{K})$, and let E_F, A_F, B_F, and Q_F, S_F, and R_F be given as in (6.4) and (6.5). Moreover, consider U and $\widehat{U}$ as in (6.10) and (6.7), respectively. Then we obtain $\big(s\widehat{\mathcal{E}}_F - \widehat{\mathcal{A}}_F\big)\widehat{Y}_F = \widehat{Z}_F\big(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\big)$ with $s\widehat{\mathcal{E}}_F - \widehat{\mathcal{A}}_F$ as in (6.6), and

$$\widehat{Y}_F := \widehat{U}^{-1}U^{-1}\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix} = \widehat{U}^{-1}\begin{bmatrix} Y_{\mu,1} \\ Y_{\mu,2} \\ Y_{x,1} \\ Y_{x,2} \\ Y_{u,F} \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m \\ 0 & 0 & 0 & I_{n_2} & B_2 \\ 0 & I_{n_1} & Q_{12}^* & \frac{1}{2}Q_{22} & -\frac{1}{2}Q_{22}B_2 + S_2 \end{bmatrix}\begin{bmatrix} Y_{\mu,1} \\ Y_{\mu,2} \\ Y_{x,1} \\ Y_{x,2} \\ Y_{u,F} \end{bmatrix} =: \begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,F} \\ \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix},$$

$$\widehat{Z}_F := \widehat{U}^{-1}U^{-1}Z.$$

Since $\operatorname{im}\widehat{Y}_F$ is $\widehat{\mathcal{E}}_F$-neutral, the space $\operatorname{im}\left[\begin{smallmatrix} Y_{\mu,1} \\ Y_{x,1} \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} 0 & -I_{n_1} \\ I_{n_1} & 0 \end{smallmatrix}\right]$-neutral and thus its dimension is at most n_1. On the other hand, since $\mathcal{V}_{{\rm sys},F} \subset \left[\begin{smallmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,F} \end{smallmatrix}\right]$ with $\mathcal{V}_{{\rm sys},F}$ as in (3.1) and $\dim \mathcal{V}_{{\rm sys},F} = n_1 + m$, it follows that $\operatorname{rank}\left[\begin{smallmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,F} \end{smallmatrix}\right] = n_1 + m$. Together with $\operatorname{rank} Y_{x,1} = n_1$, this yields $\operatorname{rank}\left[\begin{smallmatrix} Y_{x,1} \\ Y_{u,F} \end{smallmatrix}\right] = n_1 + m$. Moreover, by construction we have $\operatorname{rank}\left[\begin{smallmatrix} \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{smallmatrix}\right] = n_2$. From these facts it follows that there exists a matrix $V \in \mathrm{Gl}_{n+m}(\mathbb{K})$ such that

$$\begin{bmatrix} Y_{\mu,1} \\ Y_{x,1} \\ Y_{u,F} \\ \widehat{Y}_{x,2} \\ \widehat{Y}_{\mu,2} \end{bmatrix} V = \begin{bmatrix} X_{11} & 0 & 0 \\ I_{n_1} & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & \widetilde{Y}_{x,2} \\ 0 & 0 & \widetilde{Y}_{\mu,2} \end{bmatrix},$$

where $\operatorname{rank}\left[\begin{smallmatrix} \widetilde{Y}_{x,2} \\ \widetilde{Y}_{\mu,2} \end{smallmatrix}\right] = n_2$. From [38, Thm. 11] there exist $K_1 \in \mathbb{K}^{q\times n_1}$ and $L_1 \in \mathbb{K}^{q\times m}$ such that

$$\begin{bmatrix} 0 & -sI_{n_1} + A_{11} & B_1 \\ sI_{n_1} + A_{11}^* & Q_{11} & -Q_{12}B_2 + S_1 \\ B_1^* & -B_2^*Q_{12}^* + S_1^* & B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R \end{bmatrix}\begin{bmatrix} X_{11} & 0 \\ I_{n_1} & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -X_{11} & K_1^* \\ 0 & L_1^* \end{bmatrix}\begin{bmatrix} -sI_{n_1} + A_{11} & B_1 \\ K_1 & L_1 \end{bmatrix}, \tag{6.11}$$

where $\operatorname{rank}_{\mathbb{K}(s)}\left[\begin{smallmatrix} -sI_{n_1}+A_{11} & B_1 \\ K_1 & L_1 \end{smallmatrix}\right] = n_1 + q$, and thus (X_{11}, K_1, L_1) solves the Lur'e equation (5.11). Then, by (5.12), we obtain a solution of the Lur'e equation (5.18) which, by Theorem 5.9 (c), is simultaneously a solution of the Lur'e equation (5.1). This completes the proof. $\square$

Remark 6.3. Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with system space $\mathcal{V}_{\rm sys} \subset \mathbb{K}^{n+m}$ and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$. Let $(X, K, L) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{q\times n} \times \mathbb{K}^{q\times m}$ a solution of the Lur'e equations (5.1), and assume that $\Pi \in \mathbb{K}^{n\times n}$ is a projector with (5.16).

(a) By carefully inspecting the proof of Theorem 6.2 (see, e.g., (6.8)) we see that we can construct Y_x such that $\left[\begin{smallmatrix} Y_x \\ Y_u \end{smallmatrix}\right] \in \mathrm{Gl}_n(\mathbb{K})$. Then for $\left[\begin{smallmatrix} Y_x^- & Y_u^- \end{smallmatrix}\right] := \left[\begin{smallmatrix} Y_x \\ Y_u \end{smallmatrix}\right]^{-1}$, we obtain

$$\begin{bmatrix} Y_\mu \\ Y_x \\ Y_u \end{bmatrix}\begin{bmatrix} Y_x^- & Y_u^- \end{bmatrix} = \begin{bmatrix} X\Pi E + G_1 & G_2 \\ I_n & 0 \\ 0 & I_m \end{bmatrix}, \tag{6.12}$$

where $\operatorname{im} G_1 \subset \ker \Pi E$ and $\operatorname{im} G_2 \subset \ker \Pi E$. This yields

$$E^*\Pi^* X \Pi E = E^*\Pi^* Y_\mu Y_x^-,$$

from which a solution matrix X as well as K and L can be reconstructed.

(b) By looking at (6.9), we can perform a simple equivalence transformation to achieve

$$s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}} = \begin{bmatrix} -sE + A & B \\ K & L \end{bmatrix} \tag{6.13}$$

in (6.3).

(c) In [38, 43] it is shown how to choose deflating subspaces $\operatorname{im} Y$ which correspond to solutions of the Lur'e equation (5.1). This is achieved by a transformation of $s\mathcal{E} - \mathcal{A}$ to *even Kronecker canonical form (EKCF)* [41], which is a structured version of the Kronecker canonical form [16, 26]. In particular, it is shown that special choices of Y yield stabilizing and anti-stabilizing solutions (if they exist). Moreover, using the EKCF, further results with regard to the solution structure can be obtained [43].

(d) If $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is impulse controllable, then the projector Π can be omitted in (6.1). This is, in general, no longer the case for systems which are not impulse controllable. For instance, consider the system

$$sE - A = \begin{bmatrix} s & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = 0_{3\times 1}, \quad R = 0.$$

A simple calculation yields that (X, K, L) solves the Lur'e equation (5.1) if and only if $K = 0_{0\times 3}$, $L = 0_{0\times 1}$ and

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}$$

for some $x_{12}, x_{13}, x_{22}, x_{23}, x_{33} \in \mathbb{K}$. It can be further verified that for some solution X and all $G_1 \in \mathbb{K}^{3\times 3}$ and $G_2 \in \mathbb{K}^{3\times 1}$ with $\operatorname{im} G_1 \subset \ker E$ and $\operatorname{im} G_2 \subset \ker E$ it holds

$$\operatorname{rank}_{\mathbb{K}(s)} (s\mathcal{E} - \mathcal{A})Y = \operatorname{rank}_{\mathbb{K}(s)} \begin{bmatrix} 0 & -sE + A & B \\ sE^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix} \begin{bmatrix} XE + G_1 & G_2 \\ I_3 & 0 \\ 0 & 1 \end{bmatrix} = 4.$$

On the other hand, we have $\operatorname{rank}_{\mathbb{K}(s)} \left(s\widetilde{\mathcal{E}} - \widetilde{\mathcal{A}}\right) = 3$, so there exists no $Z \in \mathbb{K}^{7\times 3}$ such that (6.3) is stisfied. A related effect for optimal control of differential-algebraic equations has been observed in [27].

7 The linear-quadratic optimal control problem

Let a behavioral stabilizable system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subset \mathbb{K}^{n+m}$, the space of consistent differential variables $\mathcal{V}_{\mathrm{diff}} \subset \mathbb{K}^n$, and $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, $R = R^* \in \mathbb{K}^{m\times m}$ be given. We briefly discuss consequences of the presented results for the linear-quadratic optimal control problem

Minimize

$$\mathcal{J}(x, u) = \int_0^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} \mathrm{d}\tau \tag{7.1}$$

subject to $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$.

More precisely, we are interested in the functional $V^+ : E\mathcal{V}_{\mathrm{diff}} \to \mathbb{R} \cup \{-\infty\}$ with

$$V^+(Ex_0) = \inf \left\{ \mathcal{J}(x, u) \;\middle|\; (x, u) \in \mathfrak{B}_{[E,A,B]},\ Ex(0) = Ex_0 \text{ and } \lim_{t\to\infty} Ex(t) = 0 \right\}. \tag{7.2}$$

Assume that the KYP inequality (4.2) has a solution $P \in \mathbb{K}^{n\times n}$. Let $(x,u) \in \mathfrak{B}_{[E,A,B]}$. Then, by using that $\left(\begin{smallmatrix} x(t) \\ u(t)\end{smallmatrix}\right) \in \mathcal{V}_{\text{sys}}$ for almost all $t \in \mathbb{R}$, we obtain that for all $t_2 \geq t_1$ it holds

$$\begin{aligned} x(t_2)^*E^*PEx(t_2) - x(t_1)^*E^*PEx(t_1) &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)^*E^*PEx(\tau)\,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} x(\tau)^*E^*PE\dot{x}(\tau) + \dot{x}(\tau)^*E^*PEx(\tau)\,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} x(\tau)^*E^*P(Ax(\tau)+Bu(\tau)) + (Ax(\tau)+Bu(\tau))^*PEx(\tau)\,\mathrm{d}\tau \\ &= \int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau)\end{pmatrix}^* \begin{bmatrix} A^*PE+E^*PA & E^*PB \\ B^*PE & 0\end{bmatrix}\begin{pmatrix} x(\tau) \\ u(\tau)\end{pmatrix}\mathrm{d}\tau \\ &\overset{(4.2)}{\geq} -\int_{t_1}^{t_2} \begin{pmatrix} x(\tau) \\ u(\tau)\end{pmatrix}^* \begin{bmatrix} Q & S \\ S^* & R\end{bmatrix}\begin{pmatrix} x(\tau) \\ u(\tau)\end{pmatrix}\mathrm{d}\tau. \end{aligned} \tag{7.3}$$

This means that $V : E\mathcal{V}_{\text{diff}} \to \mathbb{R}$ with $V(Ex_0) = x_0^*E^*PEx_0$ is a dissipation function according to Willems' definition in [44]. In particular, we have for all $x_0 \in \mathcal{V}_{\text{diff}}$ and $(x,u) \in \mathfrak{B}_{[E,A,B]}$ with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$ that $x_0^*E^*PEx_0 \leq \mathcal{J}(x,u)$, and hence it holds

$$x_0^*E^*PEx_0 \leq V^+(Ex_0) \quad \forall x_0 \in \mathcal{V}_{\text{diff}}.$$

In particular, the solvability of the KYP inequality (4.2) implies that $V^+(Ex_0) > -\infty$ for all $x_0 \in \mathcal{V}_{\text{diff}}$.
Now consider a solution (X,K,L) of the Lur'e equation (5.1). A calculation analogous to (7.3) yields that for all $x_0 \in \mathcal{V}_{\text{diff}}$ and $(x,u) \in \mathfrak{B}_{[E,A,B]} \cap \mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})$ with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$ it holds

$$x_0^*E^*XEx_0 + \|Kx+Lu\|^2_{\mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})} = \mathcal{J}(x,u). \tag{7.4}$$

If, additionally, (X,K,L) is a stabilizing solution, then it follows from [20, Thm 6.6 (a)] that for all $\varepsilon > 0$, $x_0 \in \mathcal{V}_{\text{diff}}$, there exists some $(x,u) \in \mathfrak{B}_{[E,A,B]} \cap \mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})$ with $Ex(0) = Ex_0$, $\lim_{t\to\infty} Ex(t) = 0$, and $\|Kx+Lu\|^2_{\mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})} < \varepsilon$. This implies that the stabilizing solution defines the optimal cost functional via

$$x_0^*E^*XEx_0 = V^+(Ex_0) \quad \forall x_0 \in \mathcal{V}_{\text{diff}}.$$

The previous findings can be used to characterize the existence and structure of the *optimal control*. That is, some $(x,u) \in \mathfrak{B}_{[E,A,B]} \cap \mathcal{L}^2(\mathbb{R},\mathbb{K}^{n+m})$ with $Ex(0) = Ex_0$ and $\lim_{t\to\infty} Ex(t) = 0$ such that the infimum in (7.2) is attained at (x,u), i.e., $V^+(x_0) = \mathcal{J}(x,u)$.

We obtain from (7.4) that an optimal control fulfills $Kx(t) + Lu(t) = 0$. The latter together with Theorem (5.9) (a) implies that $(x,u) \in \mathfrak{B}_{[E,A,B]}$ is an optimal if and only if it fulfills the differential-algebraic boundary value problem

$$\begin{aligned} \Pi E\dot{x}(t) &= Ax(t) + Bu(t), \qquad Ex(0) = Ex_0, \qquad \lim_{t\to\infty} Ex(t) = 0, \\ 0 &= Kx(t) + Lu(t). \end{aligned} \tag{7.5}$$

By using (6.3), (6.12), and (6.13), we can formally write

$$\begin{bmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}t}\Pi E + A & B \\ \frac{\mathrm{d}}{\mathrm{d}t}E^*\Pi^* + A^* & Q & S \\ B^* & S^* & R \end{bmatrix}\begin{bmatrix} X\Pi E + G_1 & G_2 \\ I_n & 0 \\ 0 & I_m\end{bmatrix}\begin{pmatrix} x(t) \\ u(t)\end{pmatrix} = \begin{bmatrix} Z_\mu \\ Z_x \\ Z_u\end{bmatrix}\underbrace{\begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}t}\Pi E + A & B \\ K & L\end{bmatrix}\begin{pmatrix} x(t) \\ u(t)\end{pmatrix}}_{\equiv 0}.$$

In particular, the function $\mu(\cdot) = (X\Pi E + G_1)x(\cdot) + G_2u(\cdot)$ is part of a solution of the boundary value problem

$$\begin{bmatrix} 0 & \Pi E & 0 \\ -E^*\Pi^* & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix} \dot{\mu}(t) \\ \dot{x}(t) \\ \dot{u}(t)\end{pmatrix} = \begin{bmatrix} 0 & A & B \\ A^* & Q & S \\ B^* & S^* & R\end{bmatrix}\begin{pmatrix} \mu(t) \\ x(t) \\ u(t)\end{pmatrix} \qquad Ex(0) = Ex_0, \quad \lim_{t\to\infty} Ex(t) = 0. \tag{7.6}$$

This corresponds to the results in [33, § 3] and [31] for the case where $\left[\begin{smallmatrix} Q & S \\ S^* & R\end{smallmatrix}\right] \geq 0$.

Remark 7.1.

(a) The authors strongly believe that for a behavioral stabilizable system it holds that the Lur'e equation (5.1) has a stabilizing solutions if and only if the optimal control problem is *feasible*. That is, the optimal value function in (7.2) fulfills $V^+(Ex_0) > -\infty$ for all $x_0 \in \mathcal{V}_{\text{diff}}$. This will be subject of a forthcoming article.

(b) By using an analogous argumentation, we can see that for anti-stabilizable $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$, the anti-stabilizing solution relates to an optimal control problem on the negative time horizon.

8 Comparison to other approaches

Here we compare our results to existing approaches for linear-quadratic optimal control and generalizations of the KYP inequality to differential-algebraic systems.

8.1 Approaches based on matrix inequalities

In [17, 19], Geerts considers the optimal control problem in which the $\mathcal{L}^2$-norm of the output $y = Cx + Du$ (with $C \in \mathbb{K}^{p\times n}$, $D \in \mathbb{K}^{p\times m}$) has to be minimized. Regularity and squareness of the pencil $sE - A$ has not been assumed. Minimization of the $\mathcal{L}^2$-norm of the output $y = Cx + Du$ means that an optimal control problem (7.1) is considered in which the weight matrix is given by

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \begin{bmatrix} C^* \\ D^* \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}. \tag{8.1}$$

Note that the optimal control problem (7.1) can be written in this form if, and only if, $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \geq 0$. The main concept in [17, 19] for the analysis of the optimal control problem is the *dissipation inequality*

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq 0, \quad P = P^*. \tag{8.2}$$

The difference to the KYP inequality (4.2) is that positive semi-definiteness on whole $\mathbb{R}^{n+m}$ instead of the system space is required. Thus we see that $P \in \mathbb{K}^{n\times n}$ fulfills KYP inequality (4.2), if (8.2) holds true. Solvability analysis of the matrix inequality (8.2) in terms of properties of the Popov function (4.1) becomes obsolete, if (8.1) holds: Equation (8.1) implies that (8.2) (and thus also (4.2)) contains the solution $P = 0_{n\times n}$. Assumption (8.1) further implies that (if $sE - A$ is regular) the Popov function (4.1) is positive semi-definite for all $\mathrm{i}\omega$ with $\det(\mathrm{i}\omega E - A) \neq 0$. The existence of extremal solutions has been as well considered. It is proven in [19, Thm. 4.7] that for impulse controllable $[E, A, B]$, there exists some particular solution $P \in \mathbb{K}^{n\times n}$ of (8.2) which fulfills $E^*PE \geq E^*\widetilde{P}E$ for all other solutions $\widetilde{P} \in \mathbb{K}^{n\times n}$ of the dissipation inequality (8.2). This maximal solution has proven to express the optimal cost V^+ as in (7.2). More precisely, [19, Thm. 4.7] states that the maximal solution of the dissipation inequality (8.2) fulfills

$$V^+(x_0) = x_0^* E^* P E x_0 \quad \forall x_0 \in \mathbb{R}^n.$$

As a consequence of the above we see that, if $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is impulse controllable and behavioral stabilizable, $P \in \mathbb{K}^{n\times n}$ is a maximal solution of (8.2), and (X, K, L) is a stabilizing solution of the Lur'e equations (5.1), then $E^*XE = E^*PE$.

Note that for an indefinite weight matrix $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$, the solution set of (8.2) may be empty even though there exists some $P \in \mathbb{K}^{n\times n}$ solving the KYP inequality (4.2) [15]. This might even be the case for impulse controllable and behavioral controllable $[E, A, B] \in \Sigma_{n,m}$.

The article [11] by Camlibel and Frasca analyses *positive realness* of transfer functions $G(s) = D + C(sE - A)^{-1}B \in \mathbb{K}(s)^{p\times m}$. That is, $G(s)$ has no poles in $\mathbb{C}_+$, and $G(\lambda) + G(\lambda)^* \geq 0$ for all $\lambda \in \mathbb{C}_+$. The latter implies that the Popov function (4.1) with $Q = 0_{n\times n}$, $S = C$ and $R = D + D^*$ is positive semi-definite for all $\mathrm{i}\omega$ with $\det(\mathrm{i}\omega E - A) \neq 0$. Under the assumption of *minimality* (which includes behavioral controllability and impulse controllability) it has been shown (see eq. (23) in [11]) that positive realness is equivalent to (in our notation)

$$\begin{bmatrix} A^*PE + E^*PA & E^*PB + C^* \\ B^*PE + C & D + D^* \end{bmatrix} \geq_{\mathcal{V}_{\text{sys}}} 0, \quad P = P^* \leq 0. \tag{8.3}$$

This generalizes the famous *positive real lemma* [1] to the differential-algebraic case. This approach is more special than ours in the sense that a cost functional with particular structure is considered. The problem treated in [11] is however different in the sense that, first, the property $G(\lambda)+G(\lambda)^*\geq 0$ for all $\lambda\in\mathbb{C}_+$ has been analyzed (which only comprises $\Phi(\mathrm{i}\omega)$ for all $\omega\in\mathbb{R}$ with $\det(\mathrm{i}\omega E-A)\neq 0$) and, second, the linear matrix inequality (8.3) includes negative semi-definiteness of P. For the KYP inequality for ODE systems, the problem of existence of negative semi-definite solutions plays an important role in dissipativity analysis [44, 45].

Another approach which is related to ours has been made by BRÜLL in [8,9], where systems $[E,A,B]\in\Sigma_{n,m}$ are considered which are assumed to be *completely controllable* (a property which implies impulse controllability and behavioral controllability): The positivity of the Popov function (4.1) has been related to the existence of some $H\in\mathbb{K}^{n\times n}$ and some $J\in\mathbb{K}^{n\times m}$ such that

$$\begin{bmatrix}A^*H+H^*A+Q & A^*J+H^*B+S\\ B^*H+J^*A+S^* & B^*J+J^*B+R\end{bmatrix}\geq 0,\quad E^*H=H^*E,\quad E^*J=0. \tag{8.4}$$

By using the feedback equivalence form (2.3) and the fact that, for impulse controllable $[E,A,B]$, the system space $\mathcal{V}_{\mathrm{sys}}$ fulfills (3.3), it can be verified that $H\in\mathbb{K}^{n\times n}$ fulfills the alternative KYP inequality (4.14) if and only if there exists some $J\in\mathbb{K}^{n\times m}$ such that (H,J) fulfills (8.4). This does no longer hold true for systems which are not impulse controllable.

8.2 Approaches based on generalized algebraic Riccati equations

In [28,29], KURINA considers the optimal control problem (7.1) with positive semi-definite weight matrix $\left[\begin{smallmatrix}Q & S\\ S^* & R\end{smallmatrix}\right]$ with positive definite input weight R. The optimal value function (7.2) has been proven to fulfill $V^+(x_0)=x_0^*X^*Ex_0$ for all $x_0\in\mathbb{K}^n$, where $X\in\mathbb{K}^{n\times n}$ is a stabilizing solution of the generalized algebraic Riccati equation

$$A^*X+X^*A-(X^*B+S)R^{-1}(B^*X+S^*)+Q=0,\quad E^*X=X^*E, \tag{8.5}$$

That is, (8.5) holds true and the pencil $sE-(A-BR^{-1}(B^*X+S^*))$ has index at most one and its generalized eigenvalues are contained in $\mathbb{C}_-$. By using [10, Cor. 7 and p. 59], the existence of such a matrix X requires impulse controllability of $[E,A,B]$. MINAMINO and KATAYAMA has proven in [22] that a sufficient criterion for the existence of a stabilizing solution is $\left[\begin{smallmatrix}Q & S\\ S^* & R\end{smallmatrix}\right]\geq 0$, $R>0$, and

$$\operatorname{rank}\begin{bmatrix}-\mathrm{i}\omega E+A & B\\ Q & S\\ S^* & R\end{bmatrix}=n+m\quad\forall\,\omega\in\mathbb{R}. \tag{8.6}$$

The relation

$$\begin{aligned}&\begin{bmatrix}I_n & (X^*B+S)R^{-1/2}\\ 0 & R^{1/2}\end{bmatrix}\begin{bmatrix}A^*X+X^*A-(X^*B+S)R^{-1}(B^*X+S^*)+Q & 0\\ 0 & I_m\end{bmatrix}\begin{bmatrix}I_n & 0\\ R^{-1/2}(B^*X+S^*) & R^{1/2}\end{bmatrix}\\ &=\begin{bmatrix}A^*X+X^*A+Q & X^*B+S\\ B^*X+S & R\end{bmatrix}\end{aligned}$$

shows that, if X is a stabilizing solution of (8.5), then (X,K,L) with $K=R^{-1/2}(B^*X+S^*)$ and $L=R^{1/2}$ is a stabilizing solution of the Lur'e equations (5.15). Using this, we see that the assumption $\left[\begin{smallmatrix}Q & S\\ S^* & R\end{smallmatrix}\right]$ can be dropped in the above depiction: If $R>0$ and $X\in\mathbb{K}^{n\times n}$ is a stabilizing solution of (8.5), then $V^+(x_0)=x_0^*X^*Ex_0$ for all $x_0\in\mathbb{K}^n$.

Solvability analysis of this type of generalized algebraic Riccati equations has been treated by KATAYAMA, KAWAMOTO and TAKABA in [23,24]: The *generalized algebraic Riccati equation*

$$A^*X+X^*A+Q+X^*RX=0,\quad E^*X=X^*E,$$

is considered for E, A, Q, $R\in\mathbb{K}^{n\times n}$ with $Q=Q^*$ and $R=R^*$. A solution $X\in\mathbb{K}^{n\times n}$ is called *stabilizing*, if the pencil $-sE+A+RX$ has index at most one and its generalized eigenvalues are contained in $\mathbb{C}_-$ (which requires impulse controllability of $[E,A,R]\in\Sigma_{n,n}$. It has been proven that a stabilizing solution can be found via deflating subspaces of *Hamiltonian matrix pencils*. This is very much related to our results in Section 6: Indeed, the Hamiltonian matrix pencil corresponding to the algebraic Riccati equation (8.5) can be obtained by elementary row and column operations on the even matrix pencil (6.1) (note that $\Pi = I_n$ in the impulse controllable case). It has been further proven in [24] that solvability of the generalized algebraic Riccati equation requires the solvability of the so-called *quadratic matrix equation* $A_0^* X_0 + X_0^* A_0 + Q_0 + X_0^* R_0 X_0 = 0$, where A_0, Q_0, R_0 are matrices which are obtained from A, Q, and R by multiplication with basis matrices of $\ker E$ and $\ker E^*$. This extra condition is artificial and does not have an interpretation in terms of the dynamics of the underlying differential-algebraic equation.

Lewis considers in [31] the optimal control problem (7.1) with $R \geq 0$, $Q \geq 0$ and $S = 0$. It has been furthermore assumed that the index of $sE - A$ is bounded from above by one. The boundary value problem (7.6) and its connection to the optimal control problem has been analyzed. In the case $R > 0$, the optimal cost is shown to be expressible by means of the stabilizing solution (that is, $sE - A + BR^{-1}B^*X$ has index at most one and the generalized eigenvalues are in $\mathbb{C}_-$) of the *generalized algebraic Riccati equation*

$$A^*XE + E^*XA - E^*XBR^{-1}B^*XE + Q = 0, \quad X = X^*, \tag{8.7}$$

By an argumentation as for the Riccati equation (8.5), we obtain that, if $X \in \mathbb{K}^{n\times n}$ is a stabilizing solution of (8.7), then (X, K, L) with $K = R^{-1/2}(B^*X + S^*)$ and $L = R^{1/2}$ is a stabilizing solution of the Lur'e equation (5.1).

The big disadvantage of the approaches by generalizations of the algebraic Riccati equation is that the input weight matrix R has to be invertible. To circumvent this problem, Bender, Laub [3] and Mehrmann [33] apply a coordinate transformation of the state such that the matrix in front of the derivative reads $E = \operatorname{diag}(I_r, 0_{(n-r)\times(n-r)})$, and thereafter they extract an inherent ODE optimal control problem. The assumptions that $[E, A, B]$ is impulse controllable, (8.6), and $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right] \geq 0$ have been made. We note that this approach is - in theory - based on a transformation into feedback equivalence form (2.3). The assumptions in [3, 33] yield that, in the notation of Lemma 5.4, it holds $n_3 = 0$ and $B_2^*Q_{22}B_2 - B_2^*S_2 - S_2^*B_2 + R$. This yields that the Lur'e equation (5.9) can be transformed into an algebraic Riccati equation.

9 Conclusions and Outlook

In this paper we have proven a differential-algebraic version of the Kalman-Yakubovich-Popov (KYP) lemma. It states that the solvability of a certain matrix inequality is sufficient for the positive definiteness of the Popov function on the imaginary axis. Necessity holds true in the case where the differential-algebraic equation (DAE) is behavioral controllable or, in the case where the Popov function has full rank, it fulfills the weaker condition of sign-controllability. We have further studied the rank-minimizing solutions of the KYP inequality: These fulfill a differential-algebraic version of the Lur'e equation. Particular solutions of this equation, namely the stabilizing and anti-stabilizing ones, have been considered. We have proven that these solutions are extremal in terms of definiteness on some subspace. Implications for the linear-quadratic optimal control problem have been presented.

The equations extends generalized algebraic Riccati equations which have been treated in a couple of articles. We have shown that the Lur'e equations admit weaker solvability conditions. In particular, we have dropped several common assumptions such as impulse controllability or positive semi-definiteness of the weighting matrix.

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