Some Applications of Optimal Control in Sustainable Fishing in the Baltic Sea

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Abstract

Issues related to the implementation of dynamic programming for optimal control of a three-dimensional dynamic model (the fish populations management problem) are presented. They belong to a class of models called Lotka-Volterra models. The existence of bionomic equilibria will be considered. The problem of optimal harvest policy is then solved for the control of various classes of its behaviour. Therefore the focus will be the optimality conditions by using the Bellman principle. Moreover, we consider a different form for the optimal value of the control vector, namely the feedback or closed-loop form of the control. Academic examples are studied in order to demonstrate the proposed methods.

Keywords: Optimal control problems, maximum principle, piecewise constant optimal control, Bellman principle

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1. The problem

Currently the fish populations in the Baltic Sea have many problems, which are mainly caused by human influence. Some fish species are catched too much. The fundamental risk of overfishing is that a stock (occurrence of species in a given region) is so decimated that the natural regeneration ability is not given and at worst the species die out. The Living Planet Index for marine species of the WWF shows an average decrease of 14 % between 1970 and 2005 (see Living Planet Report 2008). The overfishing is the main cause apart from possible environmental factors (climate change, pollutants, etc).

Therefore, the goal of the Baltic Sea fishermen must be conscientious, by the policy prescribed regulations and the advance (such as from International Council for the Exploration of the Sea) to protect the Baltic Sea fauna deal. A responsible management must reduce the fishing effort to an environmentally acceptable level and call for the cooperation among the participating countries. This is of utmost importance, since the economic value of the catches depend on the stock and the biodiversity of the Baltic Sea.

Several interacting species are modeled, which inhabit in a common habitat with limited resources. So, a dynamic system is to be studied, which depends on several states and controls (e.g. the number of fishing boats). A typical question for such systems is to find a controller that regulates the system in a desired target. In many applications a cost functional is to be optimized, this is usually a functional of the state trajectory and the controls of the system. The profit of a sustainable fishing industry should be maximized without disappearance of the species.

In this paper necessary (and sometimes sufficient) optimality conditions are derived. Numerical methods are obtained from the optimality conditions in order to calculate (approximately) optimal controls.

2. Optimal control problems

Whenever a state function depending on the time is described by an ordinary differential equation which depends on the control variable, it is called a control system of ordinary differential equations. Optimal control is related to the development of space flight and military researches beginning from the 1950s. We can find the applications of the control theorie in economics, in chemistry or even in population dynamics. The general task of optimal control is defined as follows:
Let $\Omega \subset \mathbb{R}^m$ be a nonempty (often convex and closed) control region. Let

g, q, f be given smooth functions:

$$
\begin{aligned}
&q : \mathbb{R}^{n+1} \to \mathbb{R} \\
&f : \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n \\
&g : \mathbb{R} \times \mathbb{R}^n \times \Omega \to \mathbb{R}.
\end{aligned}
$$

A continuous and piecewise continuously differentiable function $x(\cdot) : \mathbb{R} \to \mathbb{R}^n$ (state function) as well as a piecewise continuous (or piecewise constant) function $u(\cdot) : \mathbb{R} \to \Omega$ (control function) are called admissible, if the ODE

$$
\begin{aligned}
\dot{x}(t) &= f(t, x(t), u(t)), \quad t_0 \le t \le T \\
x(t_0) &= x_0
\end{aligned}
$$

is valid. We are looking for admissible pairs $(x(\cdot), u(\cdot))$, which maximize an objective (cost) functional of Bolza type:

$$
J(u(\cdot)) = \int_{t_0}^{T} g\left(t, x(t), u(t)\right) dt + q(T, x(T)) \to \max_{u(\cdot)} \tag{1}
$$

Often the optimal control can be calculated by methods using the Pontryagin maximum principle or by solving the Hamilton-Jacobi-Bellman equation.

3. Extended Lotka-Volterra models with m populations

A logistic model of development for a two-population system can be written in the following form (see [13]). Let be $\varepsilon_1, \varepsilon_2$ growth coefficients, γ_1, γ_2 the phagos coefficients and K_1, K_2 given numbers (capacities or logistical terms). We denote the population sizes as x_1 and x_2.
The differential equations for the development of the populations are

$$
\begin{cases}
\dot{x}_1(t) = x_1 \left[\varepsilon_1 \left(1 - \dfrac{x_1(t)}{K_1}\right) - \gamma_1 x_2\right] \\
\dot{x}_2(t) = -x_2 \left[\varepsilon_2 \left(1 - \dfrac{x_2(t)}{K_2}\right) - \gamma_2 x_1\right]
\end{cases} .
$$

We denote generally:
ε_i are growth coefficients, γ_{ij} are the phagos coefficients of the population i with respect to the population j and K_i are logistical terms.
We denote the control of the fish populations $u_i(t)$ (it can be a regulation of the fishing, e.g. the number of the fishing boats if $u_i(t) \in \mathbb{N}$), p_i are fish

prices (per ton), r_i are catch proportionalities. Therefore, the development of m populations can be described by a generalized system

$$\dot{x}_i(t) = \varepsilon_i x_i(t) \cdot \left(1 - \frac{x_i(t)}{K_i}\right) - \sum_{j=1}^{m} \gamma_{ij} \frac{x_i(t)}{K_i} \frac{x_j(t)}{K_j} - u_i\,(t)\,r_i d \cdot \frac{x_i(t)}{K_i},$$

where $x_i(0) = x_{i0}$ are given for $i = 1, \ldots, m$.
The objective function (the profit) is to be maximized:

$$J\,(u) = \int_0^T \left\{ \sum_{i=1}^{m} p_i u_i(t) r_i d \cdot \frac{x_i(t)}{K_i} - cd \cdot \sum_{i=1}^{m} u_i(t) \right\} e^{-\delta t} dt \to \max_{u(\cdot)}$$

under the restrictions $0 \leq u_i(t) \leq u_i^{\max},\ i = 1, \ldots m,\ 0 \leq t \leq T$.
c are the cutter costs per day and d is the number of days in which we catch. If we calculate the present value of future profits, we consider a discount rate $e^{-\delta t}$. This plays an important role in economic models.

4. Bellman's principle

A key aspect of dynamic programming is the Bellman principle. The basic idea is to calculate the optimal solutions of many small subproblems and then to compose these subsolutions to a suitable global optimal solution. It was formulated in 1957 by Bellman.

„An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must be an optimal policy with regard to the state resulting from the first decision.“[15, 370-371]

This idea can be used to derive a necessary and sufficient condition.

We consider here two forms of the optimal controls of (1), namely the open-loop form and the closed-loop form. The closed-loop form $\hat{u}(t, x)$ gives the optimal value of the control vector as a function of the time and the current state.

The form of the optimal control vector derived via the necessary conditions is called open-loop. However, even though the closed-loop $\hat{u}(\cdot, \cdot)$ and open-loop $u^*(\cdot)$ controls differ in form, they yield identical values for the optimal control at each date of the planning horizon. It follows $\hat{u}(t, x^*(t)) = u^*(t)$. The open-loop form gives the optimal value of the control vector as a function of the time and the initial values of the state vector. The closed-loop form of the optimal control is a decision rule, for it gives the optimal value of the control for any current period and any admissible state in the current period that may arise. In contrast, the open-loop form of the optimal control