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Research Article

The Uniqueness of Analytic Functions on Annuli Sharing Some Values

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The purpose of this paper is to deal with the shared values and uniqueness of analytic functions on annulus. Two theorems about analytic functions on annulus sharing four distinct values are obtained, and these theorems are improvement of the results given by Cao and Yi.

1. Introduction

In this paper, we will study the uniqueness problem of analytic functions in the field of complex analysis and adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained (see [1–3]).

We use $\mathbb{C}$ to denote the open complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane, and $\mathbb{X}$ to denote the subset of $\mathbb{C}$. For $a \in \overline{\mathbb{C}}$, we say that $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities (ignoring multiplicities) in $\mathbb{X}$ (or $\mathbb{C}$) if two meromorphic functions f and g share the value a CM (IM) in $\mathbb{X}$ (or $\mathbb{C}$). In addition, we also use $f = a \rightleftharpoons g = a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that f and g share the value a CM in $\mathbb{X}$ (or $\mathbb{C}$), $f = a \Leftrightarrow g = a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that f and g share the value a IM in $\mathbb{X}$ (or $\mathbb{C}$), and $f = a \Rightarrow g = a$ in $\mathbb{X}$ (or $\mathbb{C}$) to express that $f = a$ implies $g = a$ in $\mathbb{X}$ (or $\mathbb{C}$).

In 1929, Nevanlinna (see [4]) proved the following well-known theorem.

Theorem 1.1 (see [4]). *If f and g are two nonconstant meromorphic functions that share five distinct values a_1, a_2, a_3, a_4, and a_5 IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.*

After his theorem, the uniqueness theory of meromorphic functions sharing values in the whole complex plane attracted many investigations (see [2]). In 2003, Zheng [5] studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. There were many results in the field of the uniqueness with shared values in the complex plane and angular domain, see ([5–12]). The whole complex plane $\mathbb{C}$ and angular domain all can be regarded as simply connected region. *Thus, it is interesting to consider the uniqueness theory of meromorphic functions in the multiply connected region.* Here, we will mainly study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. By the doubly connected mapping theorem [13] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case the homothety $z \mapsto z/\sqrt{rR}$ reduces the given domain to the annulus $\{z : 1/R_0 < |z| < R_0\}$, where $R_0 = \sqrt{R/r}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto 1/z$ in two cases.

In 2005, Khrystiyanyn and Kondratyuk [14, 15] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [16]). We will show the basic notions of the Nevanlinna theory on annuli in the next section. In 2009, Cao et al. [17, 18] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna's famous five-value theorem as follows.

Theorem 1.2 (see [18, Theorem 3.2]). *Let f_1 and f_2 be two transcendental or admissible meromorphic functions on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \le +\infty$. Let a_j $(j = 1, 2, 3, 4, 5)$ be five distinct complex numbers in $\overline{\mathbb{C}}$. If f_1, f_2 share a_j IM for $j = 1, 2, 3, 4, 5$, then $f_1(z) \equiv f_2(z)$.*

Remark 1.3. For the case $R_0 = +\infty$, the assertion was proved by Kondratyuk and Laine [16].

From Theorem 1.2, we can get the following results easily.

Theorem 1.4. *Under the assumptions of Theorem 1.2, if f_1, f_2 are two transcendental or admissible analytic functions on annulus $\mathbb{A}$ and f_1, f_2 share a_j IM for $j = 1, 2, 3, 4$, then $f_1(z) \equiv f_2(z)$.*

In fact, we will prove some general theorems on the uniqueness of analytic functions on the annuli sharing four values in this paper (see Section 3), and these theorems improve Theorem 1.4.

2. Basic Notions in the Nevanlinna Theory on Annuli

Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \le +\infty$. We recall the classical notations of the Nevanlinna theory as follows:

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f) \log R,$$
$$m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(Re^{i\theta}\right) \right| d\theta, \qquad T(R, f) = N(R, f) + m(R, f), \tag{2.1}$$

where $\log^+ x = \max\{\log x, 0\}$ and $n(t, f)$ is the counting function of poles of the function f in $\{z : |z| \le t\}$. We here show the notations of the Nevanlinna theory on annuli. Let

$$N_1(R,f) = \int_{1/R}^{1} \frac{n_1(t,f)}{t} dt, \qquad N_2(R,f) = \int_{1}^{R} \frac{n_2(t,f)}{t} dt,$$
$$m_0(R,f) = m(R,f) + m\left(\frac{1}{R}, f\right), \qquad N_0(R,f) = N_1(R,f) + N_2(R,f), \tag{2.2}$$

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function f in $\{z : t < |z| \le 1\}$ and $\{z : 1 < |z| \le t\}$, respectively. The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(R,f) = m_0(R,f) - 2m(1,f) + N_0(R,f) \tag{2.3}$$

and has the following properties.

Proposition 2.1 (see [14]). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \le +\infty$. Then,*

(i) $T_0(R, f) = T_0(R, 1/f)$,

(ii) $\max\{T_0(R, f_1 \cdot f_2), T_0(R, f_1/f_2), T_0(R, f_1 + f_2)\} \le T_0(R, f_1) + T_0(R, f_2) + O(1)$.

By Proposition 2.1, the first fundamental theorem on the annulus $\mathbb{A}$ is immediately obtained.

Theorem 2.2 (see [14] (the first fundamental theorem)). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \le +\infty$. Then*

$$T_0\left(R, \frac{1}{f-a}\right) = T_0(R,f) + O(1) \tag{2.4}$$

for every fixed $a \in \mathbb{C}$.

Khrystiyanyn and Kondratyuk also obtained the lemma on the logarithmic derivative on the annulus $\mathbb{A}$.

Theorem 2.3 (see [15] (lemma on the logarithmic derivative)). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $R_0 \le +\infty$, and let $\lambda > 0$. Then,*

(i) *in the case $R_0 = +\infty$,*

$$m_0\left(R, \frac{f'}{f}\right) = O(\log(RT_0(R,f))) \tag{2.5}$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\lambda-1} dR < +\infty$;

(ii) *if* $R_0 < +\infty$, *then*

$$m_0\left(R, \frac{f'}{f}\right) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{2.6}$$

for $R \in (1, R_0)$ *except for the set* Δ'_R *such that* $\int_{\Delta'_R} dR/(R_0 - R)^{\lambda-1} < +\infty$.

We denote the deficiency of $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with respect to a meromorphic function f on the annulus $\mathbb{A}$ by

$$\delta_0(a, f) = \delta_0(0, f - a) = \liminf_{r \to R_0} \frac{m_0(r, 1/(f - a))}{T_0(r, f)} = 1 - \limsup_{r \to R_0} \frac{N_0(r, 1/(f - a))}{T_0(r, f)} \tag{2.7}$$

and denote the reduced deficiency by

$$\Theta_0(a, f) = \Theta_0(0, f - a) = 1 - \limsup_{r \to R_0} \frac{\overline{N}_0(r, 1/(f - a))}{T_0(r, f)}, \tag{2.8}$$

where

$$\begin{aligned} \overline{N}_0\left(r, \frac{1}{f - a}\right) &= \overline{N}_1\left(R, \frac{1}{f - a}\right) + \overline{N}_2\left(R, \frac{1}{f - a}\right) \\ &= \int_{1/R}^{1} \frac{\overline{n}_1(t, 1/(f - a))}{t} dt + \int_{1}^{R} \frac{\overline{n}_2(t, 1/(f - a))}{t} dt \end{aligned} \tag{2.9}$$

in which each zero of the function $f - a$ is counted only once. In addition, we use $\overline{n}_1^{k)}(t, 1/(f - a))$ (or $\overline{n}_1^{(k}(t, 1/(f - a))$) to denote the counting function of poles of the function $1/(f - a)$ with multiplicities $\leq k$ (or $> k$) in $\{z : t < |z| \leq 1\}$, each point counted only once. Similarly, we can give the notations $\overline{N}_1^{k)}(t, f)$, $\overline{N}_1^{(k}(t, f)$, $\overline{N}_2^{k)}(t, f)$, $\overline{N}_2^{(k}(t, f)$, $\overline{N}_0^{k)}(t, f)$, and $\overline{N}_0^{(k}(t, f)$.

Khrystiyanyn and Kondratyuk [15] first obtained the second fundamental theorem on the the annulus $\mathbb{A}$. Later, Cao et al. [18] introduced other forms of the second fundamental theorem on annuli as follows.

Theorem 2.4 (see [18, Theorem 2.3] (the second fundamental theorem)). *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be q distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Let $\lambda \geq 0$. Then,*

(i) $(q - 2)T_0(R, f) < \sum_{j=1}^{q} N_0(R, 1/(f - a_j)) - N_0^{(1)}(R, f) + S(R, f)$,

(ii) $(q - 2)T_0(R, f) < \sum_{j=1}^{q} \overline{N}_0(R, 1/(f - a_j)) + S(R, f)$,

where

$$N_0^{(1)}(R, f) = N_0\left(R, \frac{1}{f'}\right) + 2N_0(R, f) - N_0(R, f'), \tag{2.10}$$

and (i) in the case $R_0 = +\infty$,

$$S(R, f) = O(\log(RT_0(R, f))) \tag{2.11}$$

for $R \in (1, +\infty)$ except for the set Δ_R such that $\int_{\Delta_R} R^{\lambda-1} dR < +\infty$; (ii) if $R_0 < +\infty$, then

$$S(R, f) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right) \tag{2.12}$$

for $R \in (1, R_0)$ except for the set Δ'_R such that $\int_{\Delta'_R} dR/(R_0 - R)^{\lambda-1} < +\infty$.

Definition 2.5. Let $f(z)$ be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. The function f is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$ provided that

$$\limsup_{R \to \infty} \frac{T_0(R, f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty, \tag{2.13}$$

or

$$\limsup_{R \to R_0} \frac{T_0(R, f)}{-\log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty, \tag{2.14}$$

respectively.

Thus, for a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$, $S(R, f) = o(T_0(R, f))$ holds for all $1 < R < R_0$ except for the set Δ_R or the set Δ'_R mentioned in Theorem 2.3, respectively.

3. The Main Theorems and Some Lemmas

Now we show our main results, which improve Theorem 1.4.

Theorem 3.1. *Let f, g be two analytic functions on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and let $a_j \in \mathbb{C}$ $(j = 1, 2, 3, 4)$ be four distinct values. If f and g share the two distinct values a_1, a_2 CM in $\mathbb{A}$ and $f = a_3 \Rightarrow g = a_3$ in $\mathbb{A}$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{A}$, and f is transcendental or admissible on $\mathbb{A}$, then $f(z) \equiv g(z)$.*

Theorem 3.2. *Under the assumptions of Theorem 3.1, with CM replaced by IM, we have either $f(z) \equiv g(z)$ or*

$$f \equiv \frac{a_3 g - a_1 a_2}{g - a_4}, \tag{3.1}$$

and $a_1 + a_2 = a_3 + a_4$, a_3, and a_4 are exceptional values of f and g in $\mathbb{A}$, respectively.

Remark 3.3. It is easily seen that Theorems 3.1 and 3.2 are improvement of Theorem 1.4.

To prove the above theorems, we need some lemmas as follows.

Lemma 3.4. *Let f, g be two distinct analytic functions on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$, and let $a_j \in \mathbb{C}$ $(j = 1,2,3,4)$ be four distinct complex numbers. If $f = a_j \Rightarrow g = a_j$ in $\mathbb{A}$ for $j = 1,2,3,4$ and if f is transcendental or admissible on $\mathbb{A}$, then g is also transcendental or admissible.*

Proof. By the assumption of Lemma 3.4 and applying Theorem 2.4(ii), we can get

$$\begin{aligned} 3T_0(R,f) &\leq \sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) + S(R,f) \\ &\leq \sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{g-a_j}\right) + S(R,f) \\ &\leq 4T_0(R,g) + S(R,f). \end{aligned} \tag{3.2}$$

Therefore

$$T_0(R,f) \leq 4T_0(R,g) + o(T_0(R,f)) \tag{3.3}$$

holds for all $1 < R < R_0$ except for the set Δ_R or the set Δ'_R mentioned in Theorem 2.3, respectively. Then, from Definition 2.5, we get that g is transcendental or admissible on $\mathbb{A}$. □

Lemma 3.5. *Suppose that f is a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p$ $(a_0 \neq 0)$ be a polynomial of f with degree p, where the coefficients a_j $(j = 0,1,\ldots,p)$ are constants, and let b_j $(j = 1,2,\ldots,q)$ be q $(q \geq p+1)$ distinct finite complex numbers. Then,*

$$m_0\left(R, \frac{P(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) = S(R,f). \tag{3.4}$$

Proof. From Theorem 2.3 and the definition of $m_0(R,f)$, transcendental and admissible function, we can get this lemma by using the same argument as in Lemma 4.3 in [2]. □

Lemma 3.6. *Let f, g be two distinct analytic functions on the annulus $\mathbb{A} = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Suppose that f and g share a_1, a_2 IM in $\mathbb{A}$, and $f = a_3 \Rightarrow g = a_3$ in $\mathbb{A}$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{A}$, and $a_j \in \mathbb{C}$ $(j = 1,2,3,4)$ are four distinct finite complex numbers. If f is a transcendental or admissible function on $\mathbb{A}$, then g is also transcendental or admissible, and*

(i) $T_0(R,g) = 2T_0(R,f) + S(R)$,

(ii) $T_0(R,f-g) = 3T_0(R,f) + S(R)$;

(iii) $T_0(R,f) = \overline{N}_0(R,1/(f-a_3)) + \overline{N}_0(R,1/(f-a_4)) + S(R)$,

(iv) $T_0(R,f) = \overline{N}_0(R,1/(f-a_j)) + S(R)$, $j = 1,2$,

(v) $T_0(R,g) = \overline{N}_0(R,1/(g-a_j)) + S(R)$, $j = 3,4$,

(vi) $T_0(R,f') = T_0(R,f) + S(R)$, $T_0(R,g') = T_0(R,g) + S(R)$,

where $S(R) := S(R,f) = S(R,g)$.

Proof. By the assumption of this lemma and by Theorem 2.4(ii), we have $T_0(R,f) \le 3T_0(R,g) + S(R,f)$ and $T_0(R,g) \le 3T_0(R,f) + S(R,g)$. Thus, we can get $S(R,f) = S(R,g)$.

Let

$$\eta := \frac{f'g'(f-g)}{(f-a_3)(f-a_4)(g-a_1)(g-a_2)}. \tag{3.5}$$

From the conditions of this lemma, we can get that η is analytic on $\mathbb{A}$ and $\eta \not\equiv 0$ unless $f \equiv g$. By Lemma 3.5, we have $m_0(R,\eta) = S(R,f) + S(R,g) = S(R)$. Thus, we can get $S(R,\eta) = S(R)$.

Since f,g are two nonconstant analytic functions on annulus $\mathbb{A}$ and share a_1, a_2 IM in $\mathbb{A}$ and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{A}$, again by Theorem 2.4, we have

$$3T_0(R,f) \le \sum_{j=1}^{4} \overline{N}_0\left(R, \frac{1}{f-a_j}\right) + S(R,f), \tag{3.6}$$

$$\le \overline{N}_0\left(r, \frac{1}{f-g}\right) + S(R,f) = T_0(R,f-g) + S(R,f), \tag{3.7}$$

$$\le T_0(R,f) + T_0(R,g) + S(R), \tag{3.8}$$

$$T_0(R,g) \le \overline{N}_0\left(R, \frac{1}{g-a_1}\right) + \overline{N}_0\left(R, \frac{1}{g-a_2}\right) + S(R,g), \tag{3.9}$$

$$= \overline{N}_0\left(R, \frac{1}{f-a_1}\right) + \overline{N}_0\left(R, \frac{1}{f-a_2}\right) + S(R), \tag{3.10}$$

$$\le 2T_0(R,f) + S(R). \tag{3.11}$$

From (3.8) and (3.11), we can get (i), and from (3.7), (3.8), and (i), we can get (ii), and from (3.6), (3.8), (3.10), (3.11), and (i), we can get (iii). Thus, we can deduce that (iv) and (v) hold easily from (3.6)–(3.11) and (i)–(iii). Now, we will prove that (vi) holds as follows.

First, we can rewrite (3.5) as

$$f = f'\frac{g'}{\eta(g-a_1)(g-a_2)} + \frac{f'g'(a_3f + a_4f - a_3a_4 - fg)}{\eta(f-a_3)(f-a_4)(g-a_1)(g-a_2)}. \tag{3.12}$$

From (3.12) and Lemma 3.5, we can get $m_0(R,f) \le m_0(R,f') + S(R,f)$. Since f is analytic on $\mathbb{A}$, we have

$$T_0(R,f) \le T_0(R,f') + 2m(1,f') - 2m(1,f) + S(R,f). \tag{3.13}$$

From the fact that f is transcendental or admissible, we have

$$T_0(R,f) \le T_0(R,f') + S(R,f) + O(1) = T_0(R,f') + S(R,f). \tag{3.14}$$

On the other hand, since $m_0(R,f') = m_0(R,f(f'/f)) \le m_0(R,f) + m_0(R,f'/f) + O(1)$, from Theorem 2.3, we have $m_0(R,f') \le m_0(R,f) + S(R,f)$. Thus, we can get

$$T_0(R,f') \le T_0(R,f) + S(R,f) + O(1) = T_0(R,f) + S(R,f). \tag{3.15}$$

From (3.14), (3.15) and the fact that f is transcendental or admissible, we can get $T_0(R,f') = T_0(R,f) + S(R,f)$. Similarly, we can get $T_0(R,g') = T_0(R,g) + S(R,g)$.

Thus, we complete the proof of this lemma. □

4. The Proof of Theorem 3.1

Suppose $f \not\equiv g$. By the assumption of Theorem 3.1, we can get the conclusions (i)–(vi) of Lemma 3.6 and that g is transcendental or admissible on $\mathbb{A}$. Set

$$\begin{aligned} \psi_1 &:= \frac{f'(f-a_3)}{(f-a_1)(f-a_2)} - \frac{g'(g-a_3)}{(g-a_1)(g-a_2)}, \\ \psi_2 &:= \frac{f'(f-a_4)}{(f-a_1)(f-a_2)} - \frac{g'(g-a_4)}{(g-a_1)(g-a_2)}. \end{aligned} \tag{4.1}$$

By Lemma 3.4, we can get

$$m_0(R,\psi_i) = S(R,f) + S(R,g) = S(R), \quad i = 1,2. \tag{4.2}$$

Moreover, we can prove $N_0(R,\psi_i) = O(1)$ $(i = 1,2)$. In fact, the poles of ψ_i in $\mathbb{A}$ only can occur at the zeros of $f - a_j$ and $g - a_j$ $(i,j = 1,2)$ in $\mathbb{A}$. Since f, g share a_1, a_2 CM in $\mathbb{A}$, we can see that if $z_0 \in \mathbb{A}$ is a zero of $f - a_j$ with multiplicity $m(\ge 1)$, then $z_0 \in \mathbb{A}$ is a zero of $g - a_j$ $(j = 1,2)$ with multiplicity $m(\ge 1)$. Suppose that

$$f - a_j = (z - z_0)^m \alpha_j(z), \qquad g - a_j = (z - z_0)^m \beta_j(z), \tag{4.3}$$

where $\alpha_j(z), \beta_j(z)$ are analytic functions in $\mathbb{A}$ and $\alpha_j(z_0) \ne 0$, $\beta_j(z_0) \ne 0$ $(j = 1,2)$; by a simple calculation, we have

$$\psi_i(z_0) = K\left(\frac{\alpha_j'(z_0)}{\alpha_j(z_0)} - \frac{\beta_j'(z_0)}{\beta_j(z_0)}\right) \quad (i,j = 1,2), \tag{4.4}$$

where K is a constant. Therefore, we can get that ψ_i $(i = 1,2)$ are analytic in $\mathbb{A}$. Thus, from (4.2), we can get $T_0(R,\psi_i) = m_0(R,\psi_i) + O(1) = S(R)$ $(i = 1,2)$.

If $\psi_i \not\equiv 0$, $i = 1, 2$, then we have

$$\begin{aligned} \overline{N}_0\left(R, \frac{1}{f - a_3}\right) &\leq N_0\left(R, \frac{1}{\psi_1}\right) \leq T_0(R, \psi_1) + S(R, f) + O(1) = S(R), \\ \overline{N}_0\left(R, \frac{1}{f - a_4}\right) &\leq N_0\left(R, \frac{1}{\psi_2}\right) \leq T_0(R, \psi_2) + S(R, f) + O(1) = S(R). \end{aligned} \tag{4.5}$$

From (4.5) and Lemma 3.6(iv), we have $T_0(R, f) \leq S(R)$. Thus, since f, g are transcendental or admissible functions on $\mathbb{A}$, that is, f and g are of unbounded characteristic, and from the definition of $S(R)$, we can get a contradiction.

Assume that one of ψ_1 and ψ_2 is identically zero, say $\psi_1 \equiv 0$; then we have

$$\overline{N}_0^{(2}\left(R, \frac{1}{g - a_4}\right) = \overline{N}_0^{(2}\left(R, \frac{1}{f - a_4}\right). \tag{4.6}$$

From (3.5), we can see that $g(z_1) = a_4$ implies that $f(z_1) = a_4$ for such $z_1 \in \mathbb{A}$ satisfying $\eta(z_1) \neq 0$. Since $T_0(R, \eta) = S(R)$, we have

$$\overline{N}_0^{1)}\left(R, \frac{1}{g - a_4}\right) = \overline{N}_0^{1)}\left(R, \frac{1}{f - a_4}\right) + S(R). \tag{4.7}$$

From (4.6) and (4.7), we can get

$$\overline{N}_0\left(R, \frac{1}{g - a_4}\right) = \overline{N}_0\left(R, \frac{1}{f - a_4}\right) + S(R). \tag{4.8}$$

Similarly, when $\psi_2 \equiv 0$, we can get

$$\overline{N}_0\left(R, \frac{1}{g - a_3}\right) = \overline{N}_0\left(R, \frac{1}{f - a_3}\right) + S(R). \tag{4.9}$$

From (4.8), (4.9), and Lemma 3.6(i), (v), we can get

$$2T_0(R, f) = \overline{N}_0\left(R, \frac{1}{f - a_3}\right) + S(R) \tag{4.10}$$

or

$$2T_0(R, f) = \overline{N}_0\left(R, \frac{1}{f - a_4}\right) + S(R). \tag{4.11}$$

Since f, g are transcendental or admissible functions on the annulus $\mathbb{A}$, we can get a contradiction again.

Thus, we complete the proof of Theorem 3.1.

5. The Proof of Theorem 3.2

Suppose that $f \not\equiv g$. By Theorem 2.4(ii) and the fact that f is transcendental or admissible on $\mathbb{A}$, we have

$$\begin{aligned}2T_0(R,f)+\overline{N}_0\left(R,\frac{1}{g-a_4}\right) &\le \overline{N}_0\left(R,\frac{1}{f-a_1}\right)+\overline{N}_0\left(R,\frac{1}{f-a_2}\right)+\overline{N}_0\left(R,\frac{1}{f-a_3}\right)\\ &\quad+\overline{N}_0\left(R,\frac{1}{g-a_4}\right)+S(R,f)\\ &\le \overline{N}_0\left(R,\frac{1}{f-g}\right)+S(R,f)\\ &\le T_0(R,f)+T_0(R,g)+S(R,f)+S(R,g).\end{aligned} \tag{5.1}$$

Therefore, we have

$$T_0(R,f)+\overline{N}_0\left(R,\frac{1}{g-a_4}\right)\le T_0(R,g)+S(R,f)+S(R,g). \tag{5.2}$$

Similarly, we have

$$T_0(R,g)+\overline{N}_0\left(R,\frac{1}{f-a_3}\right)\le T_0(R,f)+S(R,g)+S(R,f). \tag{5.3}$$

From (5.2) and (5.3), we can see that $T_0(R,f)=T_0(R,g)+S(R,f)+S(R,g)$, and

$$\overline{N}_0\left(R,\frac{1}{f-a_3}\right)=S(R,f)+S(R,g),\qquad \overline{N}_0\left(R,\frac{1}{g-a_4}\right)=S(R,f)+S(R,g). \tag{5.4}$$

Thus, from (5.2), (5.3), and the definition of $S(R)$, we can get that g is also transcendental or admissible on $\mathbb{A}$ when f is transcendental or admissible on $\mathbb{A}$.

From (5.1)–(5.4), we can also get

$$2T_0(R,f)=\overline{N}_0\left(R,\frac{1}{f-a_1}\right)+\overline{N}_0\left(R,\frac{1}{f-a_2}\right)+S(R). \tag{5.5}$$

From (5.5), we can see that "almost all" of zeros of $f-a_i$ $(i=1,2)$ in $\mathbb{A}$ are simple. Similarly, "almost all" of zeros of $g-a_i$ $(i=1,2)$ in $\mathbb{A}$ are simple, too. Let

$$\begin{aligned}\varphi_1 &:= \frac{(a_1-a_3)f'(f-a_2)}{(f-a_1)(f-a_3)}-\frac{(a_1-a_4)g'(g-a_2)}{(g-a_1)(g-a_4)},\\ \varphi_2 &:= \frac{(a_2-a_3)f'(f-a_1)}{(f-a_2)(f-a_3)}-\frac{(a_2-a_4)g'(g-a_1)}{(g-a_2)(g-a_4)}.\end{aligned} \tag{5.6}$$

By Lemma 3.5, we can get that $m_0(R,\varphi_i) = S(R)$ $(i = 1,2)$. Since f, g share a_1, a_2 IM in $\mathbb{A}$ and from (5.2), we have $N_0(R,\varphi_i) = S(R)$ $(i = 1,2)$. Therefore, we can get $T_0(R,\varphi_i) = S(R)$, $(i = 1,2)$.

If $\varphi_1 \not\equiv 0$, then we have $\overline{N}_0(R, 1/(f-a_2)) \le \overline{N}_0(R, 1/\varphi_1) = S(R)$. Thus, from (5.5), we can get a contradiction easily. Similarly, when $\varphi_2 \not\equiv 0$, we can get a contradiction, too. Hence, φ_1, φ_2 are identically equal to 0. Then, we have $(\varphi_1 - \varphi_2)/(a_1 - a_2) \equiv 0$, that is,

$$\frac{f'}{f-a_3} - \frac{g'}{g-a_4} - \frac{f'}{f-a_1} + \frac{g'}{g-a_1} - \frac{f'}{f-a_2} + \frac{g'}{g-a_2} \equiv 0, \tag{5.7}$$

which implies that

$$\frac{f-a_3}{g-a_4} \cdot \frac{(g-a_1)(g-a_2)}{(f-a_1)(f-a_2)} \equiv c, \tag{5.8}$$

where c is a nonzero constant. Rewrite (5.8) as

$$g^2 - \left(a_1 + a_2 - \frac{c\gamma(f)}{f-a_3}\right) g + a_1 a_2 + \frac{c a_4 \gamma(f)}{f-a_3} \equiv 0, \tag{5.9}$$

where $\gamma(f) := (f-a_1)(f-a_2)$. The discriminant of (5.9) is

$$\Delta(f) = \left(a_1 + a_2 - \frac{c\gamma(f)}{f-a_3}\right)^2 - 4\left(a_1 a_2 + \frac{c a_4 \gamma(f)}{f-a_3}\right) := \frac{\wp(f)}{(f-a_3)^2}, \tag{5.10}$$

where

$$\wp(z) := \left((a_1+a_2)(z-a_3) - c\gamma(z)\right)^2 - 4a_1a_2(z-a_3)^2 - 4ca_4\gamma(z)(z-a_3) \tag{5.11}$$

is a polynomial of degree 4 in z. If a is a zero of $\wp(z)$ in $\mathbb{A}$, obviously $a \ne a_3$. Then, from (5.9), $f(z) = a$ implies that

$$g(z) = \frac{1}{2}\left(a_1 + a_2 - \frac{c\gamma(a)}{a-a_3}\right) := b. \tag{5.12}$$

Set

$$\begin{gathered}
\phi_1 := \frac{f'g'(f-g)}{(f-a_1)(g-a_2)(f-a_3)(g-a_4)}, \\
\phi_2 := \frac{f'g'(f-g)}{(f-a_2)(g-a_1)(f-a_3)(g-a_4)}, \\
\phi = \frac{\phi_2}{\phi_1} = \frac{(f-a_1)(g-a_2)}{(f-a_2)(g-a_1)}.
\end{gathered} \tag{5.13}$$

By Lemma 3.5, we can get $m_0(R,\phi_i) = S(R)$ $(i = 1,2)$. And by a simple calculation, we can get $N_0(R,\phi_i) = S(R)$ $(i = 1,2)$. Then we have $T_0(R,\phi_i) = S(R)$ $(i = 1,2)$, thus we have $T_0(R,\phi) = S(R)$.

Assume that f is not a Möbius transformation of g; then ϕ is a nonconstant function. Since

$$\begin{aligned}\wp(a_1) &= ((a_1+a_2)(a_1-a_3))^2 - 4a_1a_2(a_1-a_3)^2 = (a_1-a_3)^2(a_1-a_2)^2 \neq 0,\\ \wp(a_2) &= ((a_1+a_3)(a_2-a_3))^2 - 4a_1a_2(a_2-a_3)^2 = (a_2-a_3)^2(a_1-a_2)^2 \neq 0.\end{aligned} \tag{5.14}$$

From $a \neq a_i$ $(i = 1,2)$ and (5.8), we can get

$$\overline{N}_0\left(R,\frac{1}{f-a}\right) \le \overline{N}_0\left(R,\frac{1}{\phi-\xi}\right) \le T_0(R,\phi) = S(R), \tag{5.15}$$

where $\xi = (a-a_1)(b-a_2)/(a-a_2)(b-a_1)$. Since f is transcendental or admissible analytic in $\mathbb{A}$, by Theorem 2.4(ii) and (5.4), we can get

$$T_0(R,f) \le \overline{N}_0\left(R,\frac{1}{f-a_3}\right) + \overline{N}_0\left(R,\frac{1}{f-a}\right) + S(R) = S(R). \tag{5.16}$$

Since f,g are transcendental or admissible functions on $\mathbb{A}$, from the above inequality, we can get a contradiction. Therefore, we can get that f is a Möbius transformation of g on $\mathbb{A}$. Since f,g are transcendental or admissible functions on $\mathbb{A}$, by a simple calculation, we can get easily that $a_1 + a_2 = a_3 + a_4$ and

$$f \equiv \frac{a_3g - a_1a_2}{g - a_4}. \tag{5.17}$$

Furthermore, a_3, a_4 are Picard exceptional values of f and g in $\mathbb{A}$, respectively.

Thus, we complete the proof of Theorem 3.2.

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